DOCUMENT RESUME

ED 411 127 SE 056 279

AUTHOR	Damarin, Suzanne K., Ed.; Shelton, Marilyn, Ed.
TITLE	Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (7th, Columbus, Ohio, October 2-5, 1985).
INSTITUTION	International Group for the Psychology of Mathematics Education. North American Chapter.
PUB DATE	1985-10-00
NOTE	393p.
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	MF01/PC16 Plus Postage.
DESCRIPTORS	Academically Gifted; Affective Behavior; *Algebra; *Arithmetic; Beliefs; Cognitive Development; *Computer Uses in Education; Educational Research; Elementary Secondary Education; *Geometry; *Mathematics Education; Metacognition; Problem Solving; Sex Differences; Spatial Ability; *Teacher Education

ABSTRACT

This proceedings from the annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education includes the following topics and authors: area measurement (C. B. Beattys, C. A. Maher); error patterns (H. C. Bebout); formative evaluation (J. C. Bergeron, N. Herscovics, N. Nantais); interactive computer environment (J. L. Bohren); emotion and cognition (L. Brandau); meaning (C. A. Brown, T. J. Cooney); misconceptions research (J. Confrey, A. Lipton); variable (C. L. Crook); number sentences (E. De Corte, L. Verschaffel); teacher attitude (L. J. DeGuire); arithmetical schemes (B. A. Eshun); everyday problems (R. Even); teaching models (E. Filloy, T. Rojano); problem solving (M. L. Frank); similarity concepts (A. Friedlander, G. Lappan, W. M. Fitzgerald); word problems in high school (C. Gaulin, A. El Boudali); heuristics (G. A. Goldin); problem solving (G. A. Goldin, J. H. Landis); metacognition (L. C. Hart, K. Schultz); proportional reasoning (P. Heller, T. R. Post, M. J. Behr); algebra (N. Herscovics, L. Chalouh); multiplication (R. Howell, B. Sidorenko, J. Jurica); word problems (J. J. Kaput, J. L. Schwartz, J. S. Poholsky); algebraic equations (C. Kieran); linguistic model (D. Kirshner); learned helplessness (P. Kloosterman); teaching strategies (J. R. Kolb, W. Truman); error patterns (V. L. Kouba); authoring languages (R. Lesh); problem solving (C. A. Maher); children's heuristics (C. A. Maher, A. Alston); chronometric analysis (J. Mestre, W. Gerace, A. Well); predicting achievement (M. R. Meyer, E. Fennema); preservice attitudes (D. Miller); computers and mathematical thinking (J. M. Oprea); decimal concepts (D. T. Owens); preservice views (J. Owens, E. Henderson); iteration (M. K. Prichard); inclusive solutions (S. K. Reed); research methodologies (D. Reinking); neuropsychological research (L. J. Sheffield); geometry and computers (M. Shelton); auditory perception (G. B. Shirk, C. O. DeFosse); diagrams (M. A. Simon); anxiety (R. G. Underhill, J. R. Becker); arithmetic books (J. van den Brink); symbolic algebra computers (R. H. Wenger); and representational schemes (G. B. Willis, K. C. Fuson). Symposia topics are: research framework for concept and principle learning (J. L. Kolb; L. Sowder; L. V. Stiff; P. S. Wilson); multifaceted cognitive domain: implications for teaching (D. Buerk; R. L. Dees; M. A. Farrell; J. L. McDonald); and Logo and mathematics learning

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(S. B. Ludwig, T. E. Kieren; J. Olive; P. W. Thompson; W. Watkins, G. Brazier; P. S. Wilson). (MKR)

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ED 411 127

PROCEEDINGS OF THE
SEVENTH ANNUAL MEETING

PME-NA

NORTH AMERICAN CHAPTER
OF THE
INTERNATIONAL GROUP
FOR THE
PSYCHOLOGY OF MATHEMATICS
EDUCATION

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COLUMBUS, OHIO

2-5 OCTOBER 1985

PROCEEDINGS OF THE
SEVENTH ANNUAL MEETING

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NORTH AMERICAN CHAPTER
OF THE
INTERNATIONAL GROUP
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PSYCHOLOGY OF MATHEMATICS
EDUCATION

EDITED BY
SUZANNE K. DAMARIN
MARILYN SHELTON

COLUMBUS, OHIO
2-5 OCTOBER 1985

PREFACE

The papers prepared for the seventh annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education reflect the numerous ways in which mathematics education has evolved over the decade since the organization was founded in Karlsruhe, West Germany. Especially apparent is the diversity of Psychological studies that are relevant to new research in Mathematics Education.

While cognitive studies have been the predominant area of psychological influence in past meetings, the present volume reflects the importance of other areas of Psychology as well. Studies reported here include many which focus upon variables from the affective domain; not only are studies of attitudes numerous, but so, also, are studies of the relationships among content-related beliefs and mathematical behavior. Studies based upon new theories concerning the relationships between physiological variables and learning are also represented.

The influence of modern technology upon mathematics education and related research is also evident in these pages. More than a dozen papers report on investigations in which the computer played a significant role in instruction; while some of these studies investigated the effects of computer mediation of traditional topics in mathematics others examine the utility of computer programming instruction as a means for teaching thinking processes which, in the past, have been associated with formal mathematics instruction.

Another way in which these proceedings indicate change from past meetings is the number of papers which focus primarily upon methodological issues.

Even as these proceedings reflect change in mathematics education research, they also reflect continuity in the depth of study of important topics. Thus, while many topics are new to these meetings, the "modal paper" is related to mathematical problem solving and reflects continuing advances in this very important area of Mathematics Education research.

From the point of view of organizing these Proceedings, the changes outlined above have had interesting side-effects. Whereas previous PME-NA Proceedings have been organized into major categories, the standard divisons did not seem appropriate for the current volume. Many papers crossed categories, while others had as their primary foci topics which did not fit the traditional categories. Therefore, the papers have been arranged alphabetically by author; symposia are presented at the end of the volume. In order to compensate for, and perhaps improve upon, the categorization, a topical index has been constructed. Each paper appears somewhere in this index; some papers appear more often.

The assembly of these Proceedings would have been impossible without the thorough and thoughtful work of Elizabeth Rhyner and Marilyn Shelton, both of whom spent many, many hours on this project. Many thanks to both of them.

Suzanne K. Damarin
Columbus, Ohio
October 1985

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I

PAPERS

APPROACHES TO LEARNING AREA MEASUREMENT AND ITS RELATION TO SPATIAL SKILL

BY

Candice B. Beattys and Carolyn A. Maher
Rutgers University

ABSTRACT

National reports of children's performance cite deficiencies in understanding of area and its measurement. This paper, the first in a two-part study, reports research which examines: (1) the effectiveness of the integration of concrete embodiments with 832 children in grades 5-7 for area measurement instruction and (2) the relationship of achievement to a student's spatial skill. Results from two-way ANOVAs indicated significant differences (.01) on both the post and retention tests favoring the groups that experienced manipulative treatment. A multiple regression analysis indicated instructional treatment, pretest and spatial skill interaction to be significant (.01) predictors of achievement.

BACKGROUND

Three successive National Assessment of Educational Progress (NAEP) reports have disclosed major deficiencies in student understanding of area and its measurement (Lindquist, et al., 1983; Carpenter, et al., 1980 and 1975). The most recent NAEP report indicated that only one quarter of nine-year olds and two-thirds of thirteen year olds correctly identified the number of square units covering a region. Other research (Maher and

Normandia, 1983) has revealed similar findings. The study by Lindquist, et al., suggested that area measurement is a complex concept mastered when underlying principles are understood. Research has identified two subconcepts inherent in understanding the concept of area: the selection of an appropriate unit of measurement for use as the basis for counting the number of units of which an object is comprised and understanding the invariance property of area; that is, that area does not change with partitioning or recombination of parts of a particular surface (Hirstein, et al., 1978). Most school learning of area is based on textbook presentations. Despite minor variation, area is commonly presented by a single limited exposure to a square unit and the presentation of a formula for its calculation. Many, perhaps most, mathematics educators are committed to the view that the child's experience and construction of the measurement are fundamental to learning the concept involved. Nevertheless, research in this field appears to be inconclusive. This may be attributed to individual differences in children, the choice of teaching approaches, the nature of the embodiments employed and the differences in time allotted for instruction. Among the manipulative materials employed for area measurement are graph paper, other squared materials and geoboards. The use of geoboards to facilitate learning area concepts has been suggested in a variety of articles (Holcomb, 1980; Harkin, 1975; Schnell and Klein, 1974) but research into the effectiveness of the geoboard as a single device has not been reported. The rationale for the use of manipulatives is that children in the intermediate grades can benefit by the construction of representations of mathematical ideas and thus require access to materials that make such constructions possible (Alston and Maher, 1984).

PURPOSE

This study, the first of two parts, sought to measure the effectiveness of the use of the geoboard as a concrete embodiment in area measurement instruction, its relative effectiveness in comparison with other approaches and an examination of the relationship, if any, between student achievement in area measurement, instructional method and the student's spatial skill. Part 2, the clinical component (Maher and Beattys, in progress), identifies how children acquire various aspects of the area concept through role playing tasks and focuses on the process by which children construct understanding of area and its measurement.

DESIGN

Subjects

Of the 832 children from an urban New Jersey school district, 267 were fifth graders, 279 were sixth graders, and 286 were seventh graders. The subjects were drawn from 48 intact self-contained classrooms. Placement for each class in a particular treatment was random within each of the three grade levels.

Procedure

Variations in treatment centered on the type of instructional material employed in each group: a single concrete embodiment approach (using a geoboard), a multiple embodiment approach (using flats, transparencies, cloth, squared paper), a textbook approach using no embodiment, and a control group that received no instruction in measurement. Teachers in the three treatment groups attended three thirty minute training sessions conducted by one of the researchers. Each treatment group's presentation varied according to the information pertinent to the particular treatment. A five day experimental period was preceded by an area measurement pretest and two spatial tests: the Hidden Patterns Test and the Space Visualization Tests. The experimental period was followed by a post area measurement achievement test and six weeks later by a retention test.

Analysis

An ANOVA and multiple comparison tests were used to assess the occurrence of post and retention achievement differences among the treatment groups. A least squares regression model was used for modeling the expected posttest area achievement score as a function of: the treatment, student pretest area measurement achievement, and student Hidden Pattern and Space Visualization spatial skill measures.

RESULTS

Differences between the posttest and retention test mean area measurement achievement scores and the mean pretest scores are summarized in Figures 1 and 2 by grade level.

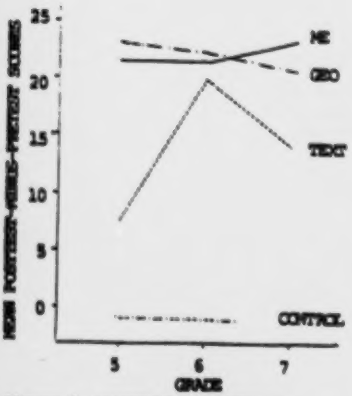

Figure 1. POSTTEST ACHIEVEMENT

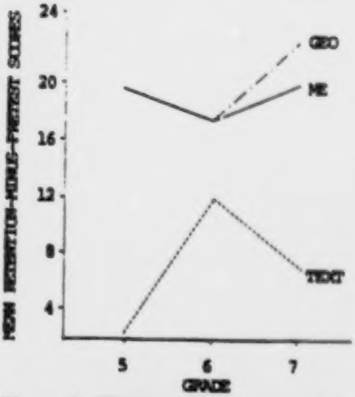

Figure 2. RETENTION TEST ACHIEVEMENT

An ANOVA and multiple regression analysis of the individual student scores support the following conclusions:

(1) Both the geoboard and the multiple embodiment approaches to learning area measurement were found superior to textbook-based instruction, regardless of the student's spatial ability.

(2) Over six weeks, advantages of manipulative-based learning over textbook-based learning increased.

(3) On the average, students who were instructed using manipulatives tended to outperform their textbook-based peers by a factor of at least two in situations requiring an application of area measurement skills.

(4) Students who scored high on the Hidden Patterns Test tended to do better with the geoboard treatment, and students who scored high on the Space Visualization Test tended to do better with the multiple embodiment treatment.

(5) Differences among grade levels 5, 6, and 7 were statistically significant but not practically substantial.

IMPLICATIONS

Results of this study, particularly the evidence from the retention test, offer convincing support to the proponents of the manipulative based instructional mode. While pretest achievement, intermediate grade level, and spatial skill also influence achievement, none of these factors minimized the effects of instructional mode. Moreover, manipulative based instruction permitted the generalization of area measurement across embodiments. Thus at the intermediate school level, better achievement for area measurement can be expected for children having experiences with manipulative based instruction.

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CHILDREN'S ERROR PATTERNS ON ADDITION AND SUBTRACTION VERBAL PROBLEMS

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Abstract

The errors of 45 first graders on addition and subtraction verbal problems were categorized as to errors of representation or errors of solution. This sample displayed different error patterns according to their abilities to concretely model verbal problem types.

Research on early number concepts has documented successful strategies that children use to solve simple addition and subtraction verbal problems (Carpenter & Moser, 1983). In addition to these successful strategies, a large number of unsuccessful solution attempts exists. As these successful strategies have provided information on children's thinking, so might their error patterns indicate additional insight.

Children's errors on abstract mathematical problems and tasks have received attention from several researchers (Brown & Burton, 1978; Brown & Vanlehn, 1980; Ginsburg, 1977; Radatz, 1979; Vanlehn, 1983); but relatively little has been written concerning children's errors on verbal mathematics problems (Briars & Larkin, 1984; Riley, 1979; Riley, Greeno, & Heller, 1983; Verschaffel, 1984). The computer simulations by Briars and Larkin (1984) and Riley et al. (1983) include error data; Briars and Larkin's model appropriately simulated 60% of the errors reported by Riley (1979), and Riley et al.'s model provided explanations for the occurrence of some errors. Verschaffel (1984), in a study of first graders' representations of verbal problems, classified errors into two categories, thinking errors and technical errors: thinking errors were those with inappropriate initial representations and technical errors were those that appeared during the calculation stage of the problem. The present study follows Verschaffel's scheme and presents children's errors as related to representation of the verbal problem or to solution or calculation following the representation.

METHOD

The subjects were 45 first graders in two classrooms in a rural Midwestern elementary school. Using nine problems from the current classification of verbal problems (Carpenter & Moser, 1984; Riley et al., 1984), the children were individually interviewed and asked to solve the problems using manipulatives. According to their use of concrete items, children were placed in three levels, Basic, Direct Modeling, and Representing. Their errors of representation and solution were noted according to problem type and according to concrete modeling level.

RESULTS AND DISCUSSION

Children's verbal problem errors were categorized as Representation Errors and Solution Errors. More specifically, Representation Errors, those that involved the modeling strategies or lack of strategies, included the following categories: Wrong Operation, Repeat Given Number, Guess, and No Attempt. Solution Errors, those that occurred after an appropriate strategy had been chosen, included the following categories: Computation, Repeat Given Number (after appropriate strategy choice), and Wrong Model (incorrectly modeling of a given element). A third category, Uncodable, contained those few errors (five responses total) that could not be categorized as Representation or Solution.

Table 1 presents for each problem type the total number of incorrect responses and the total number of Representation and Solution Errors. From the total of incorrect responses, these first graders performed as expected: they did most poorly on Start Unknown, or Change 5 and 6 verbal problems, and somewhat better on Change 3 and 4 problems. Their least number of errors were on simple problems, Change 1 and 2.

Representation and Solution Errors. Representation Errors occurred most frequently on the Change 5 and 6 problems, and somewhat less frequently on Change 3 problems. Change 4 problems followed next, with Change 1 and 2 having the least number of Representation Errors. The Repeat Given category showed a high incidence of error.

The Solution Errors in this study did not follow the same pattern as the Representation Errors. Children made as few Solution Errors on the Change 3 problem as they did on the simple Change 1 and 2 problems. The highest number of Solution Errors were on the Change 4 problems. Calculation errors were most numerous; Repeat Given occurred infrequently after correct representation.

Errors According to Modeling Level. The twelve children at the Basic or pre-direct modeling level had the most number of incorrect solutions on Change 3, 5, and 6 problems. These incorrect solutions were due to errors of representation as opposed to errors of solution. It appeared that if this group of children could represent a problem they could effect a correct solution.

The twenty-two children at the Direct Modeling level had their most difficult time on the Start Unknowns, with comparable numbers of errors in both representing and solving the problem. Their most successful representation performances were on Change 1, 2, 3, and 4 problems. In general, their errors of solution were higher than their errors of representation, except for Change 5 and 6 problems.

The elementary children at the Rerepresenting level had their highest total errors on Change 5 problems. Their errors in general were errors of solution. This group made no Representation Errors on Change 1, 2, 4, and 6 problems.

SUMMARY

Children at different concrete modeling levels produced different patterns of errors. By viewing errors according to representation of the problem or solution of the preceding representation, errors of children may be better understood. Instruction on verbal problem solving may then be better matched to the type of error.

Table 1

Verbal Problem Errors of Representation and Solution

Problem Type	Total Errors		Concrete Modeling Level								
			Basic N=12			Direct Modeling N=22			Rerepresenting N=11		
	Rep	Sol	Rep	Sol	Total	Rep	Sol	Total	Rep	Sol	Total
Change 1	0	6	0	1	1	0	5	5	0	0	0
Change 2	1	7	1	2	3	0	4	4	0	1	1
Change 3	14	7	12	0	12	1	4	5	1	3	4
Change 4	5	10	4	2	6	1	4	5	0	4	4
Change 5	21	9	11	0	11	7	6	13	3	3	5
Change 6	19	10	11	0	11	8	8	16	0	2	2

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ACKNOWLEDGMENTS

The author wishes to thank Thomas P. Carpenter and Vicky L. Kouba for their comments on this paper. The research was supported in part by the American Association of Publishers and by the Wisconsin Center for Education Research, which is supported in part by a grant from the National Institute of Education (Grant No. NIE-G-84-0008). The opinions expressed in this paper do not necessarily reflect the position, policy, or endorsement of the National Institute of Education.

FORMATIVE EVALUATION FROM A CONSTRUCTIVIST PERSPECTIVE

Jacques C. Bergeron, Université de Montréal
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Within the context of mathematics education, the current view of evaluation is open to criticism first, in terms of the rather behavioristic classification of the learning outcomes it identifies, and second, with regards to the prevalent mode of obtaining information, the written test. A constructivist approach affects our perspective of both the learner's and the teacher's role in a didactical situation, and also that of the subject matter. In such a perspective, the need for formative evaluation becomes crucial since in order to follow the student's thinking, the teacher requires feedback from him. To this effect, we have developed a new tool, the mini-interview. This paper describes an experiment investigating the problems involved in training teachers in the use of this tool for formative purposes.

THE CURRENT VIEW OF EVALUATION

As the title of their book implies, Bloom, Madaus and Hastings (1981) concern themselves with the use of Evaluation to Improve Learning. This is why they devote individual chapters to three learning related objectives of evaluation, diagnostic, summative and formative, instead of focusing on normative tests. In their view, the role of diagnostic evaluation is to determine school readiness, the proper placement of students, as well as the causes of the difficulties they may experience. In contrasting summative and formative evaluation, they identify three distinguishing characteristics. The first one has to do with purpose, the main purpose of formative evaluation being the deter-

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The research reported here is funded by grants from the Quebec Ministry of Education (FCAR- EQ-1741, EQ-2923).

We wish to thank Professor Clément Dassa of the Université de Montréal for his helpful suggestions.

mination of "the degree of mastery of a given learning task and to pinpoint the part of the task not mastered" while on the other hand, summative evaluation is directed toward a much more general assessment covering an entire course or substantial parts of it. If in the latter case a grade can be attributed to reflect the portion of the course mastered by the student, no grade is ever involved in formative tests since the anxiety they might then generate could prevent the student from perceiving them as an aid to learning. The second characteristic differentiating the two types of evaluation is the portion of course covered. Tests for formative purposes are given frequently whenever the initial instruction of a new skill or concept is completed. In contrast, summative evaluation looks at the mastery of several new skills or concepts. At the secondary level, it is used two or three times within a course as part of an overall grading system, while at the elementary level, teacher-made tests are given every four to six weeks. The third characteristic involves the level of generalization. Formative evaluation might be used to determine if the student possesses all the pre-requisites needed for a certain topic, whereas summative evaluation assesses the degree of generalization and transfer that he has achieved regarding the subject matter at hand.

As noted by Bloom et al, the main function of formative evaluation is to provide both teacher and student with feedback enabling each one to take corrective measures when needed. In designing formative tests, they first determine what new content or subject matter is involved in a new learning unit. This is then followed by an analysis of the expected "behaviors or learning outcomes" which are then classified according to a hierarchy involving six levels: knowledge of terms, facts, rules and principles, skills in using processes and procedures, ability to make translations (using one's own words, using different modes of representation), ability to make applications (using rules and principles to solve problems presented in a new context).

Serious questions can be raised regarding the appropriateness of this view of formative evaluation when judged from a constructivist perspective of mathematics education. The first one relates to the rather behavioristic classification of the above learning outcomes. For indeed, a constructivist approach takes a more global perspective and identifies the learning of mathematics with the construction of conceptual schemes.

A second question deals with the form of tests used by Bloom et al. Although they mention that "there are other ways besides paper-and-pencil tests to make inferences about student progress" (p.71), no such example is provided in their book, thus reflecting the prevalent mode of obtaining information: the written test. This kind of format encourages the student to believe that the answer is the all-important part of mathematics (Erlwanger, ,1975) and prevents him from appreciating the mathematical thinking which leads up to it. Another drawback of the written test is that it presumes a reading competency which does not yet exist in the beginning grades. Thus, for the early years, most written work is limited to numerical exercises or to problem situations represented pictorially, using conventions which the child often does not understand (Campbell,1981). However, the major deficiency of the written format is that especially with young children, it cannot inform us about the procedures they use. For instance, no written test can tell us the procedure used by the child adding 6 to 3.

A CONSTRUCTIVIST PERSPECTIVE OF MATHEMATICS EDUCATION

A constructivist approach affects our perspective of both the learner's and the teacher's role in a didactical situation, as well as that of the subject matter. The underlying assumption here is that the learner is the principal agent in the construction of his knowledge. And far from diminishing the role of the teacher, this approach necessitates on his part a much more difficult contribution. For he can no longer depend on the old belief that he can simply transmit his knowledge: he must now carefully prepare didactical situations enabling the learner to re-construct it for himself. This can be achieved if the teacher starts from the learner's existing knowledge and relies on it to help him climb up the different steps of the intended construction. For a student in such a learning situation, each step is an extension of his accrued knowledge and this endows the learning process with cognitive continuity.

The very view of what constitutes mathematics is also changed. In a constructivist perspective, mathematics cannot be reduced to the acquisition of skills and algorithms, for it is more the processes involved in mathematical thinking which are sought. Our work of the last few years clearly reflects this viewpoint. We have been concerned with developing a model which might adequately describe the processes involved in concept formation. Of course, we take "concept" in the broader sense of "conceptual scheme", that of a network of related notions, not just atoms of isolated knowledge. In mathematics, each fundamental concept (such as number, addition, function, etc.) can be described as a conceptual schema whose construction may require fairly long periods of time.

Our model identifies four stages in the construction of such a mathematical concept: the first one, the intuitive stage, involves the learner's informal knowledge and previous experience related to the given concept; the second stage concerns the acquisition of mathematical procedures which the learner can relate to his intuitive knowledge and use appropriately within the context of relevant problem situations; the third stage, that of abstraction refers to both a detachment from any concrete representation or procedure as well as the construction of invariants; the final stage, that of formalization, encapsulates the given mathematical concept into a formal definition and symbolization (Herscovics & Bergeron,1984). This model clearly illustrates the emphasis on mathematical thinking prevalent in a constructivist approach. For indeed, two specific stages are explicitly related to intellectual processes, that of intuitive thinking and that of abstraction. Of course, these are not even considered among the behavioral objectives of Bloom et al.

EVALUATION IN A CONSTRUCTIVIST PERSPECTIVE

Before introducing a new concept or continuing with a given construction, the teacher has to determine the cognitive basis on which the student can build and progress. He will then have to establish if the pre-requisite notions are present, and when needed, fill in the gaps and correct the false interpretations. This implies that the teacher has to be able to monitor each pupil's cognitive progression. While written tests are adequate in assessing mathematical skills, they can, at best, provide only indirect inferences regarding the student's thinking and reasoning. Such information is obtained more directly and more explicitly by questioning the learner. As mentioned earlier, this is particularly true for young children in the early school years, when their limited reading skills greatly restrict the value of any written test. Teachers at these levels have an urgent need for another form of evaluation.

It is in answer to such needs that we developed the MINI-INTERVIEW, an adaptation of the clinical interview methodology which takes into account the restrictions of the school environment (Nantais et al, 1983). In developing it, our objective was to provide the teacher with a tool for formative evaluation which might be integrated to his regular teaching. Since the aim was to uncover the student's thinking and reasoning, the MINI-INTERVIEW was to be used with each and every pupil. As the questioning was to take place in the classroom, this imposed a time restriction for each interview, 5 to 10 minutes, which is about the most time a teacher can devote to an individual, even under optimal class organization for independent study. Of course, this limited time and the number of children involved meant that the tasks and the questions needed to be prepared in advance, and that their scope be restricted to precise aspects of key conceptual schemes (e.g. by the end of the first grade, the mastery of the counting on procedure in addition problems).

In the past three years we have been training future and practicing elementary schoolteachers in the use of mini-interviews. This topic was dealt with as part of a university course in mathematics education in which we attempted to develop a constructivist perspective, using our model to analyze the child's construction of major arithmetical concepts such as number, the four operations, place value notation, as well as the addition and subtraction algorithms. We have been quite successful in inducing a more constructivist perception of both mathematics and the instructor's role in the teaching of mathematics. This is evidenced by our teachers' increased preoccupation with the student's mathematical thinking which for them becomes at least as important as finding the right answer. Their concern for the pupil's thinking and reasoning created a climate favorable for the introduction of the mini-interview. Not only did we present them with the theoretical background, but we also asked them to ascertain a few children's thinking about given arithmetical concepts by using previously prepared mini-interviews. We had expected that such experience would result in our teachers perceiving the mini-interview as a new tool enabling them to practice formative evaluation. To our great surprise, we discovered that quite often the mini-interview was not used for this objective, but simply as just another test to verify performance rather than as a source of feedback. The question of how to induce a formative perspective in these teachers was thus raised.

AN EXPERIMENT IN FORMATIVE EVALUATION TRAINING

The experiment we have conceived involves five first grade schoolteachers who were asked to use a mini-interview on the adding-on procedure with each pupil in their class, the interviews to be completed over a period of three weeks. This is, in some way, a feasibility study verifying if it is possible for teachers to integrate such a task to their regular classroom activity. Our second objective is to determine conditions under which they might come to view the mini-interview as a way of obtaining feedback. We cannot expect a teacher who is just starting to use the mini-interview, to perceive it immediately as a tool for formative evaluation. For he is then confronted to a host of new problems such as classroom organization and management while he attends to one child, as well as being concerned with the quality of his questioning while recording the interview. It is only gradually, as he overcomes these difficulties, that we can hope to see him grasp the formative potential of this new tool.

But to determine the conditions which might bring him to use the mini-interview for its intended purpose, we had to find ways of following each teacher's evolution over the three-week period. To achieve this, we asked every teacher to record each interview, to evaluate each child, and to keep a daily diary. The recording of each interview eliminates the need for classroom observation while preserving, as much as possible, the natural climate of the classroom. Moreover, each tape conserves accurately the content of the interview which can then be used by the teacher to evaluate his pupils. These audio-tapes can also be used by the researcher to judge the teacher's evaluation, to compare the difficulties he can detect by listening with those reported by the teacher in his diary, and to determine, to some extent, if the teacher and pupil perceive the interview as one involving formative evaluation or simply testing.

From the checklist he has filled during the interview as well as by listening to the audio-tape, the teacher had to evaluate each pupil. If in his report he indicates the kind of pedagogical intervention he envisages in order to help the child, either to overcome a learning obstacle or to follow through with the intended construction, he will then have provided us with evidence of a formative evaluation. On the other hand, if he expresses himself essentially in terms of success and failure, we will interpret it as indicating a testing approach. Finally, we expect his diary to reveal his evolution and progress in tackling the daily problems brought about by performing interviews in the classroom. The data obtained will be analyzed this Fall.

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A STUDY OF FIELD DEPENDENCE/INDEPENDENCE IN AN INTERACTIVE COMPUTER ENVIROMENT FOR GEOMETRY

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This study was conducted to determine if field independence/dependence affected one's ability to learn the transformational geometry principles of translation, rotation, and reflection from an interactive computer geometry game in which these moves were embedded. Thirteen students (grade 3 - graduate school) were observed playing the game. FDI scores were determined using the Group Embedded Figures Test. The more field independent students easily sorted out the three moves from each other, and were thus able to use them more effectively than the field dependent students.

Because geometry is a weak area in the elementary curriculum, The Technology and Basic Skills - Mathematics project (TABSMath) chose geometry as one of the four areas in which to design innovative interactive computer programs (Damarin, 1982). One of the TABSMath's geometry disks is "Funky Chicken". It is a program of experiential learning in transformational geometry, which requires the learner to use combinations of reflection, translation, and rotation to move a chicken within 4 grids to catch the elusive flies.

Field dependence/independence is a cognitive style defined as a measure of one's ability to disembed relative information from an irrelevant background and to analyze and cognitively restructure information (Witkin & Goodenough, 1981). Many studies have shown the relationships between FDI and learning (Witkin, et al., 1977). An analysis of the learning tasks of the "Funky Chicken" program indicated that the learner must be able to visualize the result of each one of a group of moves through 2-D space. In other words, students must be able to disembed each move from the total group of moves and the background grid system. The fast pace of the program increases the difficulty of disembedding individual moves from the context of the game. For this reason, it was hypothesized that the field dependent learner would take more turns than the field independent learner to master the use of translation (sliding), rotation, and reflection in this game.

One moves through a series of levels when playing the "Funky Chicken" game. On the first level the flies have one move, the second level two moves, etc. After the moves of the flies are listed, the student directs the chicken to make up to three moves to catch the flies. When all directions are entered, one sees the flies zip through their moves, and then the chicken makes its 3 moves. If it is properly directed, the chicken will land in the same quadrant as the flies and be able to eat them. If the flies are not eaten, this indicates that the moves did not do what the player thought they would. From levels 1 to 4, if one does not catch the flies, one returns to level 1. After reaching level 5, if one fails to catch the flies, one returns to level 5.

METHODS

Fourteen students, ages 8 to 30+, ranging from third grade to graduate school, were observed playing the "Funky Chicken" game. Notes were kept on the moves the flies made (computer controlled) and the moves the student intended to make and actually made to catch the flies. The game was played until the student had mastered the sliding and rotation moves or for two hours. As observations were made, it became obvious that mastering the sliding move was crucial to early success in the game (reflections do not occur at the lower levels of the game). After playing the game, each student was given the Group Embedded Figures Test (Witkin, et al., 1971) to measure their position on the field dependent/independent continuum. The author's work on other research studies with high school students had indicated that a score of 0 to 10 could be considered as relatively field dependent and 13 to 18 as field independent.

RESULTS

After observing the first few students (whose GEFT scores varied from 4 to 16), it became obvious that the more FD students were having a much harder time catching the flies because they could not figure out the results of the sliding move (translation). In addition, although they recognized that they weren't getting what they expected from a slide, they did not try to figure out what was wrong in any systematic manner. They contined to mix slides, rotations, and reflections together in their three moves. At the end of two hours, the observer gave some hints such as "watch where the head of the chicken is" or "try only slides, so you can figure out what is going on". With such direction, the more field dependent students were able to figure out what the slide move did and successfully catch the flies. The more field independent students recognized quite quickly on their own that they had to figure out what the slide was doing and were able to sort out this move from the other two types of moves, and were able to learn to catch the flies in fewer moves than the more field dependent players. A time limit of two hours was set. The most field dependent learners did not ever reach the point where the computer presented reflections (level 4). The more field independent learners were generally able to master the horizontal and vertical reflections after a few tries, because they were able to disembed the move from the context of the game. The most field independent learners reached the levels where diagonal reflections occurred and mastered this move.

DISCUSSION OF RESULTS

The most interesting question posed by this type of study is what sort of cognitive processing variables seem to be important in decision making demanded by the instruction and conversely, what instructional variables affect decision making strategies of the learner. Ausburn and Ausburn (1978) discussed the implications of a number of cognitive styles for instructional design. They proposed that cognitive processing styles, such as field dependence/independence, underlie the learner's ability or inability to build a link between the demands of a task and the learner's cognitive processing strategies. The data of this study suggest that there may be an incomplete learner/task link between the features of the cognitive processing task as modeled by the program and the encoding of the information by field dependent learners, that prevents elaboration which leads to learning. Ausburn and Ausburn suggest "compensatory supplantation" to bridge the cognitive processing gap between learner skills and task demands.

Malone (1984) suggested that challenge, fantasy and curiosity are three components that make learning fun. Green (1984) noted that good software design guaranteed the learner a "great deal of successful action". Flagg (1985) noted that themes of "chasing, fleeing, catching, or getting caught" were effective attention-grabbing characteristics for children. The "Funky Chicken" program incorporates all of the above in a carefully designed environment for problem solving. With a few compensatory supplantation segments for the field dependent learner, the program should serve most students with an exciting and successful way to learn transformational geometry concepts.

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EMOTION AND COGNITION --- EFFECTS OF FEAR OF MATHEMATICS ON LEARNING AND TEACHING

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This paper calls for research into the interrelatedness between emotion and cognition, especially as to how fear of mathematics relates to the learning of mathematics and to its subsequent effects on teaching. The ideas in this paper stem from an intense one year study of one elementary teacher who taught at a private school for about 25 children, aged 5 - 10 years old. The protocols from video and audiotapes were analyzed to explain this teacher's struggle to encourage thinking in mathematics. To have promoted the kind of thinking she wanted, the teacher would have needed to place the children in more situations for which risks could have been taken. It was proposed that the teacher was afraid that she would not have been able to handle the mistakes that could have arisen in such risky situations. Any situation involving mathematics evoked her feelings of inadequacy and fear.

The purpose of this paper is to highlight the need for mathematics education research into the interrelatedness between emotion and cognition, especially in connection to fear of mathematics. I will discuss my belief that by studying emotion AND cognition, we will gain some important insights into the learning and teaching of mathematics.

It seems to have been a long tradition in mathematics education research to separate the study of "affective" and "cognitive" variables. This has probably been due to the influence of some of the psychological research which has frowned upon studies dealing with emotions. But recently, some mathematics education researchers (such as McLeod, 1985) are calling for the integration of the affective and the cognitive. And some of the psychological research community seems to be moving in this direction, as indicated by the Jean Piaget Society's 1983 meeting titled "Emotion and Cognition", during which Jerome Bruner called for the study of the interrelation between the two.

I view this new endeavor as different from the "math anxiety" movement of some years ago however. That movement seemed to spawn many clinics whose purpose was to "treat" the math anxious person. The view seemed to be that anxiety is something to be "cured" by working with the "anxious" individual. I would argue for a different view of anxiety and fear of mathematics. I would argue that it is the SOCIAL SITUATIONS in which individuals find themselves that evoke the fear and anxiety of mathematics. We need to study those situations ---- and all the socio-cultural dimensions to them ---- to gain the most insight into "fear of mathematics".

In doing such studies, I would argue against the assumption that fear resides within an individual. That is, fear cannot be considered one of a person's attributes in the way that eye color or hair color is. If an individual shows or feels fear of mathematics, it is because the emotion is evoked by some social situation. For example, many of my students at Keene State College feel fear of mathematics. And this fear seems to be first evoked when they register for a math class. It is not that this emotion exists in them. It is in the recollection of their past experiences in mathematics classes that the same emotional reaction is now evoked. Perhaps they were called "stupid" by a previous teacher (or students) or failed enough mathematics tests (or courses) to feel inadequate. Their past experiences with the subject has given them memories of feelings of failure so that an anticipated experience evokes the same feelings. Thus registering for a math class in college is a social situation that can evoke all the unpleasant feelings associated with previous mathematics classes.

What seems especially important to me are the cognitive blocks to learning that occur due to the evokation of such fear. I had one student for whom a test would evoke such fear that her hands would shake and that problems I knew she could do would be totally muddled. On one quiz, I gave the students a "story" problem that involved area and perimeter. It was a problem which could be solved intuitively, without the usual "mathematical formulas". This one student did not do the problem because, as she later told me, she "did not trust her intuition, her own way of doing the problem". On reflection, it occurred to me that there was no reason for her to trust her own thinking in mathematics. She had had at least 14 years of failure in mathematics classes --- failing grades and being told, or inferring, she was "dumb". Why would her own strategy of solving this area and perimeter problem suddenly be worthwhile? After all her years of "failure", why should she trust her own thinking?

This year I hope to study some of my mathematics students who are studying to be elementary teachers. I am interested in their experiences with mathematics but also in the situations which evoke or lessen their fears. An important question seems to be: how can a social situation be created that will enhance students' self-worth and trust in their own mathematical thinking abilities?

Also important to me are the elementary teachers who are currently in the classroom. My interest in the interrelationship between emotion and cognition stems from my dissertation study (Brandau, 1985) of one elementary teacher and her struggle to encourage thinking in her mathematics classes. An important result of this study was my insight into her fears and how they affected her teaching.

This teacher wanted to encourage more than a "memorize-what-to-do" level of learning in her students. My analysis of the teaching and learning that occurred in her classroom showed the following. To have promoted the kind of thinking she wanted, the teacher needed to place herself and the children in situations for which more "risks" could be taken ---

"risks" that involved more trial and error and more problem solving which was open-ended. But this teacher felt inadequate about her own mathematical knowledge and abilities. For example, when she tried to write some lessons involving probability ideas, she was told by her husband that her problems for the students were much more mathematically complex than she realized. Another time she came to me to ask about the "long division algorithm" and why we do the steps in the way we do them. A student had asked her about this idea and she felt so uncomfortable with her knowledge that she could not provide an answer.

In terms of the relation between emotion and cognition, this teacher's feelings of inadequacy and fear of the students making mistakes had an effect on the way she thought about teaching, and hence on the way she taught. Because she felt uncomfortable about her ability to handle what she saw as student mistakes, she rarely put them or herself in situations for which "mistakes" could be made. That is, in trial-and-error problem solving and/or open-ended problem solving, students can devise many unusual strategies of their own --- strategies which the teacher may never have seen before. This situation alone would provoke anxiety and fear in a teacher concerned with students solving problems in a successful manner. If a "new" strategy is invented by a child, how can one be sure that it will be successful? And if one is uncomfortable with one's math knowledge, then how can there be comfortableness with teaching mathematics?

Since this teacher felt responsible for the children's learning, the consequences of them getting mathematically "lost" were high. What if she could not discern inappropriate or misleading strategies, ones that would lead the children into mathematical misconceptions? Thus it was safer for her to keep the children and herself in situations for which she felt cognitively comfortable --- situations for which she felt she could provide appropriate strategies and/or answers.

This teacher, in a sense, was in a similar situation as my students studying to be teachers. Any situation involving mathematics evoked fear and created cognitive blocks which in turn affected learning or teaching mathematics. In terms of research involving teachers in the classroom, it seems important to investigate those situations which create a comfortableness with mathematics so that the teachers' methods involve some risk taking.

In summary, research involving the interrelationship between emotion and cognition can yield some important insights into the learning and teaching of mathematics. What needs to be investigated are all dimensions of the social situations which: (a) evoke fear of mathematics and how learning and/or teaching is affected and (b) enhance the learner's or teacher's self-confidence and comfortableness with mathematics.

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THE IMPORTANCE OF MEANING AND MILIEU IN UNDERSTANDING MATHEMATICS TEACHING

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It is argued that both the contexts in which mathematics teaching occurs and the perspectives brought to the mathematics classroom by teachers and students are important considerations for understanding mathematics teaching. Research on the perspectives of teachers, students, and others involved in the classroom is reviewed and implications for research and teacher education are discussed.

Our research interests center on mathematics teachers' conceptions of both mathematics and mathematics teaching. We are concerned with the meanings teachers ascribe to classroom events and how the milieu of the classroom influences those meanings. We believe there is ample evidence (e.g., Thompson, 1984) to suggest that teachers' conceptions are related to their instructional practice, and, as Delamont (1983) has argued, "classroom processes can only be understood if their context is understood" (p. 45). However, although research on teaching has provided numerous descriptions of classroom activities, little information is available on mathematics teachers' conceptions of mathematics and mathematics teaching and classroom actions as they are related to the milieu of the classroom.

Though teachers play a fundamental role in establishing the milieu of the classroom, they are by no means the sole determiners of the environment. Our work has convinced us of the importance of considering not only the roles that teachers play, but also the roles that students and other individuals such as parents, principals, and teachers' colleagues play in the development of the classroom milieu. We will consider the perspectives of teachers, students, and significant others (parents, principals, and teachers' colleagues) and the ways in which these perspectives influence meanings and milieu in mathematics classrooms.

THE PERSPECTIVES OF TEACHERS

There is evidence that beliefs concerning subject matter influence the ways in which teachers define the classroom situation. For example, Schmidt and Buchman (1983) found that elementary teachers' beliefs about subject matter influenced the amount of time they allocated for the various subjects. Thompson (1984) found that teachers' conceptions about mathematics and mathematics teaching were a significant, albeit subtle, factor in forming their classroom actions. Though she found some inconsistencies between expressed beliefs and instructional practices, in general, teachers' conceptions were reflected in their classroom actions.

There are curricula differences which may also affect the way a teacher conceptualizes teaching and performs in the classroom. Most secondary mathematics teachers, for example, find it easier to think about, to plan for, and to teach mathematically advanced classes than general mathematics classes (Lanier, 1981). In a case study we conducted of a beginning secondary mathematics teacher, we found that the teacher's classroom actions and how he thought about them were different for different courses and that this influenced his conception of mathematics teaching.

THE PERSPECTIVES OF STUDENTS

There is a growing body of research that supports the contention that students' perspectives are also an important influencing factor in the classroom. Students bring their own individual and collective biographies to the classroom. Students may not appear to have much power, but there are many subtle ways in which they exert control. Hoyles (1982) reported that secondary students wanted security and structure when studying mathematics; they wanted to know when they had gotten it right. They insisted on being graded and viewed this as a measure of their mathematical ability. Mathematics was not seen as something that could be of interest in itself, but only as something to be mastered, something to be done. Other findings, such as Metz's (1978) report of low ability students' apparent dislike for public interaction and classroom lecture and preference for seatwork, provide additional evidence of the relatively hidden student factors with which teachers must cope.

The perspectives of teachers and students are often not congruent, which often makes consensus in the classroom difficult to achieve. Negotiations take place as teachers and students struggle to create a classroom atmosphere in which learning can take place. Krummheuer (1983) found that teachers and students have bodies of mathematical knowledge that are fundamentally different and involve qualitatively different ways of thinking about mathematics. These differences seem to inhibit the teachers' efforts to help students learn mathematics since teacher and students are often communicating at different levels and with different meanings. For example, teachers tend to operate from what Krummheuer calls an algebraic-didactic frame while students view mathematics from an algorithmic-mechanical frame. Krummheuer suggests that teachers must learn to understand the perspectives of their students and help them to understand the perspective of the teacher so that both can work together toward a common goal.

In a case study we conducted with a beginning mathematics teacher, we found that his students had conceptions of both mathematics and of how it should be taught. In general, the students believed mathematics consisted of rules and definitions to be memorized and used to solve assigned exercises. They also believed that a mathematics teacher should define terms and explain procedures carefully, working examples to show students how the exercises should be solved. Students viewed assignments as a means of practicing procedures and of indicating what they had learned. The conceptions held by the students were in conflict with those held by the teacher, who believed mathematics was a body of knowledge to be explored and appreciated. Problem solving was the essence of mathematics for this teacher and he expressed an intention to share this view with his students. His attempts to use problems in his teaching were met with indifference and sometimes resistance from the students, motivating a change both in his classroom actions and his thinking about them.

Stephens and Romberg (1985) studied Australian mathematics teachers who were attempting to use innovative materials called RIME (Reality in Mathematics Education). A reoccuring theme throughout their analysis was the difficulty teachers faced when they were asked to teach atypical content in atypical ways. It was clear from this study that a real negotiation takes place between teachers and students regarding what is judged to be acceptable mathematics and acceptable ways of teaching mathematics.

THE PERSPECTIVES OF SIGNIFICANT OTHERS

Life in the classroom is not shaped totally by the perspectives and actions of only teachers and students. Other individuals who contribute to the definition of the classroom situation include parents, principals, teaching colleagues, and in a collective sense, society at large. The literature indicates that these individuals also shape or at least provide constraints on classroom activities. Bishop and Nickson (1983), for example, hypothesized that because mathematics is a subject that is easily identifiable and generally valued by parents the teaching of mathematics is more open to criticism by parents than most other subjects. Teachers are forced to justify to parents their teaching of particular content and its relevance for the pupils' future life.

The literature and our own work has made clear to us that there are many individuals, each with unique perspectives, that affect the mathematics classroom in many ways which we do not yet fully understand. We do not know how the perspectives of various individuals interact in the complexity of the classroom setting. We do not understand why mathematics teachers do not practice in the classroom that which they have been taught in teacher education programs. We maintain that research related to understanding the perspectives of these individuals and the effects they have on the mathematics classroom can give mathematics educators additional insight into the teaching of mathematics and provide a better foundation for preparing teachers for the compexity of the mathematics classroom.

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MISCONCEPTIONS RESEARCH AND THE CLINICAL INTERVIEW

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This paper discusses the relationship between the methodology of clinical interviewing and the conceptualization of misconceptions. It suggests that with the current methods of clinical interviewing, no differentiation between students with systematic errors and weak constructive processes and students with misconceptions can be made with assurance. Plans to revise the clinical methods in response to the reconceptualization of misconceptions are reported.

INTRODUCTION

The clinical interview has been used with increasing frequency in studies of students' mathematical understanding. Its appeal was in its ability to go beyond the paper and pencil test in providing us evidence of how students are thinking about mathematical concepts, processes and reasons.

Early studies showed in dramatic ways that students were not learning what they were taught. As the work developed, researchers became aware that while the students were not learning what we expected in large measure, they were learning something; and the focus of research shifted to try to determine just what that was. While there were many reasons why students did not perform well, the pervasive and predictable qualities of the errors and students' resistance to relinquishing faulty solution strategies lead researchers to define them as misconceptions. These traits, (especially the commitment of students to their way of seeing) lead some researchers to posit that an internally consistent structure was at work behind these errors.

The clinical interview continues to be the primary data-gathering tool used in investigating the developing misconceptions research area. Its goal has been to build a map or model of the student's method of thinking by observing his/her spontaneous problem solving strategy and asking questions until a "rational" and predictive model could be abstracted from the student's statements. To achieve these goals a set of misconceptions tasks must be carefully structured. While the problems have often been original and thought-provoking, and have required the students to do more than simply apply memorized rules and methods, they have done little to clarify the relationship of misconceptions to the student's larger conceptualization of the problem; they were simply designed to "reveal" misconceptions.

It is our position that this state of the theory of misconceptions and methodology needs attention. We believe that the theory and the methods must develop dialectically if the empirical results of these studies are to aid us in improving our theoretical positions.

CONSTRUCTIVIST ASSUMPTIONS

Some of the thorniest methodological questions come from our decision to conceptualize the interview as a form of communication in which both the interviewer and the interviewee must construct meaningful interpretations of the setting. At every level of the analysis, we, as interviewers are using our *interpretation* of the inputs and of the output evidence. Inputs include the written problem, the stated goals and the interviewer's questions (we believe the student's interpretation of these inputs is likely to be quite different from ours. Evidence of outputs are verbal statements, written marks, affective signals (again, it is our interpretation of these things we use to build on). As constructivists, it is essential in our viewpoint to be aware that we are building *models* of students' thought and that these models are, in turn, derived from *interpretations* and *constructs* of inputs and output evidence. Many studies making use of the clinical interview take the interpretation of these inputs and outputs as non-problematic; they assume agreement between the student's and interviewer's interpretation.

DESIGN

The misconceptions test consisted of five problems chosen to represent: functional and algebraic manipulations, geometric area and identities, and graph and chart interpretation (see Appendix). The problems were chosen to represent a variety of concepts, and were designed to allow for multiple representations (symbol systems, graphs, tables, applications, etc). Within each problem, the multiple representations were included to allow us to explore the extent of an error pattern and its internal consistency. For example, on the first problem, a table of the exponential function $y=4^x$ with the values $x=2$ and $x=3$ was included with the problem: $4^{2.5}=$____.

In line with our concept of the interview as an intervention, we chose two interview techniques an introspective, talk-aloud approach and a more retrospective, review or retelling of work done silently. Fifteen students, chosen at random from a pre-calculus course solved these problems individually in audio-taped interviews. These students were assigned two problems as talk-aloud problems, two as review problems and were to elect one of the interview styles for the final problem. The order of the problems was systematically varied.

We tested a larger student population (n= 108) with a written version of the test to provide another perspective on the interviews. The results of the written tests were scored, and the strategies students used for each question (where apparent) noted and tallied. The scores on these tasks were distressingly low with an average correct performance on the items of 32%. Only 49% could correctly square a trinomial, for example. (The results are listed in Appendix.) A comparison revealed that the results from the two methods were largely suppporting. The interview certainly allowed us more insight into the strategies and beliefs of the students; whereas the written test sample reassured us that our interview population was representative of the group and that the problems seemed appropriate for the level of students.

SYSTEMATIC ERRORS AND WEAK CONSTRUCTIVE PROCESSES

At the time of the design of the problems, we thought of misconceptions as a system of beliefs which formed a relatively stable and internally consistent cognitive system. We expected misconceptions to be concept-specific and to be able to be analyzed into prerequisite skills, definitions, representations, related concepts and use of language. Furthermore, we expected to find students highly confident of their answers and committed to them. Our data showed that students often applied repetitive and predictable faulty strategies, but these lacked the compelling nature or internal consistency of misconceptions. This suggested the more elementary notion of systematic errors.

Systematic errors include the systematic (and inappropriate) application of familiar fragments of arguments, algorithms and definitions without any attempt to integrate accross representational systems. They are common across students and permit accurate prediction of what answers the student will give to a well-defined set of problems. They are not powerful and they are not conceptualizations. They are akin to snapshots. In contrast with those who construct full-blown misconceptions, students who make systematic errors have difficulty expressing their beliefs, their knowledge appears fragmented and isolated, and their commitment to their ideas seems minimal and hesitant, as evidenced by a drop in their voice or a quick and superficial labelling of the problem. Examples in our interviews are legion: in Problem 3, 12% of the students squared each term of the trinomial; 42% of the students did not differentiate between the square of a sum and the sum of squares; 56% multiplied indiscriminately the side and the base of a parallelogram to obtain its area in Problem 2.

In analyzing our interviews we found little evidence of more highly structured or compelling errors that would warrant the name misconception. This has led us to reconsider our definition and to give more importance to a dimension we have chosen to label the constructive process. It is in this area the dynamic of student's engagement with problem solving, where differences between successful and unsuccessful students was most prominent.

The more successful students would lead the interview, confidently and firmly; when encountering a problem they would paraphrase or reformulate it. In solving it, they would tap multiple strategies, and use language to explore ideas. Successful students confirmed, with unexpected frequency, the importance of key counter-examples as a hedge against both systematic errors. At certain junctures in problem solving where students were about to follow the less successful students in making one or another error, these counter-examples or strategic "don'ts" came to their aid, either in the form of semantic principles or syntactical rules of thumb: "...negative exponents are really positive numbers, but ... it's under a fraction, 1 over "that value". In other words, these are not negative here..." When these students were asked to review a problem they could describe unsuccessful as well as successful attempts. During the review phase they would engage problems for the second time.

In contrast, the unsuccessful problem solver would appeal to rules, algorithms and techniques. S/He would usually expect these to be produced full blown from memory. A problem that could not be solved (problem 3b) or that required exploration (problem 4) would be seen as objectionable, because in mathematics, one "does" things, and gets answers. When in doubt, the unsuccessful student tended to reach into his or her grab bag, pulling out anything which seemed related to the problem. When pressed even minimally they would drop their voices or express their intense dislike of being mistaken. They would minimize their risk by stopping at the earliest possible point when a problem was finished. The review was a routine reporting of what they did.

In summary, we are suggesting that with our initial definition of a misconception and with our current methods, we were unable to distinguish misconceptions from systematic errors. We believe that this is in part due to a lack of engagement of students in constructive processes, and this has led us to reconsider the relationship among systematic errors, weak constructive processes and misconceptions.

MISCONCEPTIONS AND METHODS REVISITED

In this last section, we will discuss the issues raised by this study that must be addressed before proceeding. No final resolution of the issues is offered at this time.

1. One interpretation might focus on the role of constructive processes in the learning of mathematics. Perhaps if students could be taught to learn mathematics the way the successful students did, the poor performance on these types of problems could be lessened considerably.

2. Another possibility to consider is that weak constructive processes mask misconceptions. If so, and if we wish to continue to pursue misconceptions, we could identify those students with strong constructive processes, and search for misconceptions in that population.

3. Alternatively, it might be more appropriate to view misconceptions in the light of Hawkins' critical barriers (1982). Critical barriers are conceptual obstacles which are critical in that they involve preconceptions, are incompatible with scientific understanding, are prevalent across individuals of different ages, educational experiences and achievement levels, are structured, and are fundamental to understanding a range of phenomena. Furthermore, Hawkins emphasized the joy and insight experienced as critical barriers are surmounted.

In integrating Hawkins' definition of critical barriers with our own observation of student performance on interview tasks, we feel that misconceptions must have a powerful underlying conceptual and psychological dimension; they must aid a student in making sense of some phenomena and must include an effective constructive process as part of problem solving behavior. This process dimension seems essential to us in order to distinguish between a misconception and a systematic error, and even more importantly perhaps, between a correct algorithmic performance and a powerful conception. Explication of an affective dimension will probably also be included.

4. Lastly, we are struggling to reconceptualize the clinical interview in order to make it sensitive to the revisions discussed above. We are convinced that it needs to be seen as an intervention, and that we need to be less timid about that intervention. If we conceive of a misconception as a powerful construction, then we believe that the methodology ought to confront the student with externally apparent contradictions and multiple representations, and therefore, to explore how resistent misconceptions are, connecting them more closely with constructive processes.

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Appendix: Tasks and Summary Results

1. Fill in the chart for the function: $y = 4^x$ for $x = (-3, -2, -1, 0, 1, 2, 3)$	correct	63%
	nearly correct	12%
	incorrect	25%
What does $4^{2.5}$ equal?	correct	44%
	incorrect	2%
What does $4^{1.5}$ equal?	correct	33%
	incorrect	67%
2. A person was given the plot of land witn the dimensions shown below and she wants to know the area. Calculate the area.	correct	5%
	identified height of or indicated that area of was needed	25%
	incorrect	70%
3. Perform the indicated operation: a) $(n^2 + 3p - 2s)^2$	correct	49%
	incorrect	51%
b) $\backslash 9 + 4x^2$	correct	28%
	incorrect	7%
4. Draw a triangle where a, b, and c are the lengths of the three sides and $a + b = c$.	correct (supported)	10%
	unable to classify	21%
	incorrect	69%
5. The Drag Race. This question required the students to interpret a graph and chart. (Carjel, C., R.Joss, G.Monk, From Problems to Calculus, Univ. of Washington, 1975, p. 2.)	correct	22%
	incomplete	19%
	incorrect	57%
	misunderstood prob.	2%

DEVELOPMENT AND VALIDATION OF AN INSTRUMENT TO ASSESS FIFTH AND SIXTH GRADE CHILDREN'S KNOWLEDGE OF VARIABLE

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ABSTRACT

The purpose of this study is to develop an instrument to assess fifth and sixth children's understanding of variable. The study was conducted in two phases. During Phase I student's answers to algebra problems were analyzed and classified into error types. During Phase II, a twenty-six item multiple choice test was developed to assess fifth and sixth grade children's knowledge of variable. Knowledge of the kinds of errors that were made, and the common incorrect responses that occurred on the tests given during Phase I, formed the basis for creating distractors for the multiple choice items. The test was administered to fifth, sixth, and seventh grade children. An item analysis, factor analysis, and multiple regression were used to determine the validity and reliability of the instrument.

Since the advent of computers in the classroom, many educators believe that there is a relationship between programming and learning about variables (Blume & Schoen, 1985; Soloway, Lochhead, & Clement, 1982; Fey et al. 1984). Traditionally, students have little experience with variables before ninth grade algebra. With the availability of computers in most elementary schools, children may be able to learn about variables at a much younger age. Currently there is no instrument available to assess an elementary student's knowledge of variable. The purpose of this study is to develop and validate an instrument to assess fifth and sixth grade children's knowledge of variable.

A twenty-six item multiple choice test was administered to two fifth, two sixth, and two seventh grade classes. Test items were categorized according to Kuchemann's (1981) different categories of letter usage. Categories were determined by deciding what would be the lowest level of understanding needed to correctly answer the problem. Since the results from Phase I indicated that items became more difficult as the category number increases, more items from the first four categories were included and only three from category five and six. Table 1 shows the number of problems in each category.

Table 1
Number of Problems in Each Category of Letter Usage

	Categories					
	Letter Evaluated	Letter Ignored	Letter as Object	Letter as Specific Unknown	Letter as Generalized Number	Letter as Variable
Number of Items	6	5	4	6	2	1

RESULTS

The means ans standard deviations for grades five, six, seven, and the total are presented in Table 2.

Table 2
Means and Standard Deviations for Grade 5, 6, 7, and Total Score

Grade	n	Mean	Standard Deviation	Standard Error
5	46	10.15	3.59	.529
6	46	11.63	3.79	.559
7	46	15.15	5.67	.836
Total	138	12.31	4.89	.417

A factor analysis showed that two factors accounted for 25.6% of the variance. Eleven test items loaded significantly on one factor and six items loaded significantly on the second factor, with a cut-off point of .375 for both factors.

Results of regression analysis found that grade level was a significant predictor of total test score, $p < .0001$. Eighteen percent of the variance in grade can be explained by the total test score.

A stepwise multiple regression performed on factor 1 and factor 2 with grade as the dependent variable found that the combination of factor 1 and factor 2 explained 20% of the variance in grade. Factor 2 alone accounted for 16% of the variance. Factor 2 is significantly related to grade level, $p < .0001$, and factors 1 and 2 combined are also significantly related to grade, $p < .0001$.

Table 3 summarizes the results of item analysis.

Table 3
Summary results of Item Analysis

	Average		Cronbach's	Number
	Item difficulty	Phi coefficient	Alpha	of items
Total	.52	.69	.81	26
Factor 1	.42	.74	.79	11
Factor 2	.71	.71	.63	6

DISCUSSION

All eleven items that loaded on factor one fit into one of Kuchemann's first three categories of letter usage. All of the problems could be solved by hints from key words, recalling a familiar bond, substituting numbers for letters, or using the letter as an object. None of the solutions involved manipulating or operating on an unknown.

All of the items that loaded on factor two fit into category four, five, or six. The solutions to the six problems that loaded on factor two required the ability to do at least one of the following:

1. to see the relationship between two unknown specified values and to find those values.
2. to see the relationships between two unspecified values
3. to manipulate and operate on an unknown
4. to regard the letter as a generalized number.

Evidence from Phase I indicates that there was a greater difference in difficulty between the first three levels and the fourth, and evidence from factor analysis in Phase II strongly suggests that there are two distinct levels of letter usage.

The first level, an elementary use of letters, does not require manipulation of the unknown factor. The solution can be found by ignoring the letter, substituting a number for the letter, or recalling a familiar bond.

The second level of letter usage requires at least an understanding of the concept of a letter representing an unknown number and the ability to operate on the letter in order to find the unknown number.

There appears to be a gap between these two levels of letter usage. This gap may be due to differences in cognitive levels. However, since the fifth and sixth grade students had very little experience with any kind of letter usage, and the seventh grade students had some pre-algebra experience, the gap could be due to learning experiences.

Further study may show that a variety of experiences of letter usage at the elementary level may provide the needed structure to bridge the gap between the two levels of letter usage. The key element between the two levels of letter usage appears to be the concept of a letter representing an unknown number and the ability to operate on the letter in order to find the unknown.

Statistical analyses tend to substantiate the assumption that the variable assessment instrument is reliable and valid. The test items that do not load on either factor need to be analyzed, and refined or deleted. Additional factor two items are needed, especially category five and category six items.

The results of this study suggest that with a few refinements the variable test is an appropriate instrument to assess fifth and sixth student's knowledge of variable.

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WRITING NUMBER SENTENCES TO REPRESENT ADDITION AND SUBTRACTION WORD PROBLEMS

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Since 1979 we are working on a research project in which we try to contribute to a theory of the development of children's ability to solve elementary arithmetic word problems. In this paper we present some results concerning the relationship between children's processes for analyzing and solving addition and subtraction word problems on the one hand, and their skill in writing number sentences to represent these problems on the other. The presentation of the results is organised around the following three topics : (1) spontaneous use of number sentences, (2) incorrect elicited number sentences, (3) correct elicited number sentences.

INTRODUCTION

In the last few years a substantial body of research has been done on the development of children's problem-solving skills and processes with respect to elementary addition and subtraction word problems (De Corte & Verschaffel, 1985a). This research has provided a fairly detailed and consistent picture of the level of difficulty of different types of elementary addition and subtraction word problems, of the variety of "informal" strategies applied by young children to solve those problems, and of the kind of errors they commit. However, the transition from these initial informal problem-solving processes to the more "formal" addition and subtraction concepts and skills taught in school is still not very well understood. One important aspect of this transition is how children gradually turn to these formal concepts and skills to represent and solve elementary arithmetic word problems. This issue was addressed in a longitudinal investigation carried out in our Center. Some findings of this study relating to problem difficulty and to children's solution strategies and errors were reported elsewhere (De Corte & Verschaffel, 1984, 1985a, 1985b). In this paper we present the main results concerning the relationship between children's processes for analyzing and solving elementary addition and subtraction word problems on the one hand, and their skill in writing number sentences to represent these problems on the other. Before discussing these findings we give a brief description of the design of our longitudinal investigation.

DESIGN AND TECHNIQUES

During the school year 1981-1982 we collected empirical data on children's representation and solution processes with respect to elementary addition and subtraction word problems. Thirty first graders were individually interviewed three times during the school year : at the very beginning in September, in January and at the end of June. Each time they were administered eight word problems : four addition and four subtraction problems :

Addition problems (2)

Change 1	Pete had 3 (5) apples; Ann gave Pete 5 (7) more apples; how many apples does Pete have now ?
Change 6	Pete had some apples; Pete gave 3 (5) apples to Ann; now Pete has 5 (7) apples; how many apples did Pete have in the beginning ?
Combine 1	Pete has 3 (5) apples; Ann has 7 (9) apples; how many apples do Pete and Ann have altogether ?
Compare 3	Pete has 3 (5) apples; Ann has 6 (8) more apples than Pete; how many apples does Ann have ?

Subtraction problems

Change 2	Pete had 6 (12) apples; he gave 2 (4) apples to Ann; how many apples does Pete have now ?
Change 3	Pete had 3 (5) apples; Ann gave Pete some more apples; now Pete has 10 (14) apples; how many apples did Ann give to Pete ?
Combine 2	Pete has 3 (5) apples; Ann has also some apples; Pete and Ann have 9 (13) apples altogether; how many apples does Ann have ?
Compare 1	Pete has 3 (5) apples; Ann has some more apples than Pete; Ann has 8 (12) apples; how many apples does Ann have more than Pete ?

The word problems were read aloud by the interviewer. With respect to each problem, the children were asked to perform the following tasks : (1) to retell the problem, (2) to solve it, (3) to explain and justify their solution strategy, (4) to build a material representation of the story with puppets and blocks, and (5) to write a matching number sentence. Task 5 was administered only during the second and the third interview, when the children had already received formal instruction on addition and subtraction. The individual interviews were videotaped. The data were submitted to a quantitative and a qualitative analysis.

RESULTS

Spontaneous use of number sentences

One of the most striking results of our study is that - although during the school year the children were explicitly taught to do so - the typical first grader did not formulate spontaneously an incomplete numerical sentence (e.g. 9-3=. or 3+.=9) as a step in the solution of a word problem (see also Carpenter, Hiebert & Moser, 1983; Lindvall & Ibarra, 1978). During the second and the third interview, no one of the eight problems was solved more than two and five times respectively using a written number sentence. During both interviews only one and three children respectively applied this "number sentence writing"-strategy more or less systematically, i.e. for at least half of the problems. The explanation for this finding is rather straigthforward. Beginning elementary school children seem to solve word problems by constructing some kind of external or internal representation of the essential elements and relations in the problem text; then they select a quantitative action to determine the unknown in that problem representation (De Corte & Verschaffel, 1985b). For these children writing a number sentence as an intermediate step between the construction of a representation of the problem situation and the selection of the quantitative action to solve the task is by no means necessary nor helpful to find the answer more quickly or more efficiently. On the contrary, due to their unfamiliarity with the formal symbols and rules of school mathematics, translating their understanding of the word problem into a numerical sentence can lead to difficulties that interfere with these children's spontaneous solution processes.

Incorrect elicited number sentences

When a word problem was solved correctly without spontaneously stating a formal equation, the child was asked to generate a matching number sentence. An analysis of the reactions on this task shows that for every problem a considerable number of children failed to formulate an appropriate number sentence for a word problem they were able to solve. Similar data were reported by Carpenter, Hiebert & Moser (1983) and by Lindvall & Ibarra

(1978). A detailed description of the different erroneous number sentences for the distinct problem types is given elsewhere (Verschaffel, 1984). Here we only discuss some data with respect to the two problem types that elicited most incorrect sentences, namely change 6 and compare 1.

During the second and the third interview more than one third of the children who had solved the change 6 problem correctly, were unable to write an appropriate number sentence. A closer look at these children's protocols suggests that for many of them the failure was due to the ambiguity of the sentence-writing task. Indeed, number sentences can fulfil two different functions with respect to word problems : they can be used either as a formal mathematical representation of the semantic relations between the known and the unknown quantities involved in the problem, or as a mathematical notation of the operation that should be or has been performed on the two given numbers to find the solution of the problem. Sometimes the same number sentence can fulfil both functions, like in the following example. Suppose a child is given the problem "Pete had 6 apples; he gave 2 apples to Ann; how many apples does Pete have now ?" and solves it by decrementing 6 by 2. In this case the number sentence 6-2=. represents the semantic structure of the problem as well as the arithmetic operation performed by the child. In other cases both aspects have to be expressed by different number sentences. For example, consider our change 6 problem ("Pete had some apples; he gave 3 apples to Ann; now Pete has 5 apples; how many apples did Pete have in the beginning ?" being solved by adding the two given numbers. The numerical sentence .-3=5 represents the semantic structure of this problem, but the arithmetic operation applied by the child matches either the number sentence 5+3=. or 3+5=. Especially with respect to our change 6 problem, several children failed on the sentence-writing task, because they tried to combine both functions into one single equation, which lead them into an unsurmountable conflict : on the one hand they acknowledged that the numerical sentence should contain a minus sign, because the word problem mentioned that Pete gave away 3 apples; on the other hand they thought that the number sentence should comprise a plus sign, because they had added the two given quantities to solve the problem.

Another remarkable finding is the great number of failures on the compare 1 problem ("Pete has 3 apples; Ann has some more apples; Ann has 8 apples; how many apples does Ann have more than Pete ?"). About half of the children solving that problem correctly, were unable to write an appropriate number sentence. The most common incorrect responses were : 3+8=5, 3+8=11, 3-8=5, 3...8=5, and "no answer". This may at least partially be due to the type of mathematics program used to teach these children arithmetic. First, in the beginning of that program, the arithmetic operations of addition and subtraction are introduced using concrete change and combine situations; it provides almost no experience in comparing the relative size of two sets (compare situations). Second, once the children have obtained a certain competence in writing and solving addition and subtraction number sentences, the program offers almost no opportunity to apply these skills to compare word problems. Therefore, it is not surprising that the children had considerable difficulties in connecting their intuitive understanding and their informal solution strategies of compare problems to the formal mathematical symbols and rules they were taught in school.

Correct elicited number sentences

We finally discuss some findings concerning the types of valid number sentences for the addition and subtraction word problems.

Three addition problems elicited only canonical addition sentences (i.e. a+b=. or b+a=.) (3): change 1, combine 1 and compare 3. This is not at all surprising. Indeed, children who try to formulate a numerical sentence that corresponds to the semantic structure of these three types of word problems as well as those seeking to express the nature of the operation performed, have to apply such a canonical equation. Interestingly, a considerable number of these canonical sentences started with the larger given quantity in spite of the fact that in the verbal text of the problem the smaller quantity was mentioned first; they were all generated by children who had solved the word problem using a strategy that starts with the larger given number (Verschaffel, 1984). Apparently these children had interpreted the "sentence-writing task" as a request to formulate the number sentence that expresses the arithmetic operation performed to find the solution of the word problem. Although the fourth addition word problem - change 6 - also elicited a great number of canonical addition sentences, non-canonical equations (.-a=b) were generated by a significant minority. As explained before, the number sentence expressing the semantic structure of a change 6 problem and that corresponding to the operation commonly performed to solve the problem can be quite different.

With respect to the subtraction word problems, only the change 2 problem elicited a substantial number of canonical subtraction sentences (a-b=.); for the three other problems - change 3, combine 2 and compare 1 - almost no subtraction sentences were generated. This is also not surprising, taking into account the structure of these word problems on the one hand, and the kind of strategies the children used to solve them on the other. Contrary to the change 2 problem, the structure of these subtraction word problems is expressed most appropriately by a non-canonical addition sentence (a+.=b). Moreover, while the change 2 problem was solved mostly with direct subtractive strategies (i.e. strategies in which the answer is found by subtracting the smaller given number from the larger one), the other subtraction problems elicited almost exclusively indirect additive strategies (i.e. strategies in which the child determines what quantity the smallest given number must be added with to obtain the larger one) (Verschaffel, 1984).

NOTES

(1) L. Verschaffel is a Senior Research Assistant of the National Fund for Scientific Research, Belgium.

(2) The numbers in brackets were used during the third interview.

(3) The symbols a and b represent the first and the second given number in the word problem.

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AN ATTITUDE SURVEY OF MATHEMATICS TEACHERS IN ELEMENTARY GRADES

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A 40-item attitude survey was given to 259 elementary teachers as part of a larger survey about variables related to the teaching of mathematics. Factor analysis of the results indicated 5 factors among the attitude items: Enjoyment of Teaching Mathematics, Ease in Learning Mathematics, What Mathematics is, Students as Learners of Mathematics, and How to Teach Mathematics.

The quality of mathematics learning depends on a variety of variables, including affective variables within the student and within the teacher. Reyes (1984) reviewed the effects on learning of affective variables within the student. Such research leads to the question of the effects on learning of affective variables within the teacher. The present paper reports on part of a study of such variables among teachers in the elementary grades and their effects on student achievement.

The study is based on data collected from all teachers of mathematics (K-12) in one school district (see DeGuire, 1985), with 259 of the respondents on the elementary level. The teachers were surveyed about the following areas:

1) their mathematics backgrounds,
2) their mathematics education backgrounds,
3) the amount of class time they give to mathematics,
4) how they use the instructional time in mathematics,
5) their confidence in teaching various topics in mathematics,
6) their attitudes and beliefs about themselves as learners of mathematics, about themselves as teachers of mathematics, about students as mathematics learners, and about what mathematics is.

Also, the results on the mathematics portions of the California Achievement Tests (CAT) for grades 1, 2, 3, and 6 were obtained. These results are composite for each grade in each building

The present paper examines the structure of the 40-item attituce section of the survey for the elementary teachers. Further analyses will examine relationships among the attitude variables, the confidence variables, the use-of-instructional-time variables, and the student achievement variables.

Method

The Subjects

The respondents to the survey were all the teachers who teach mathematics in the elemetnary grades (K-5/6) in a single district. Within the district, some 6th-grade classes are in middle schools (grades 6 to 8); other 6th-grade classes are in elementary schools (grades K to 6). Only 6th-grade teachers in elementary schools were included in the analysis reported here. The district contains 24 schools (6 high schools, 5 middle schools and junior high schools, and 13 elementary schools) and covers approximately 800 square miles. Its student population is from mid-to-lower middle class, mostly rural families. It serves approximately 17,000 students. Its teacher population is drawn mainly from nearby small cities and towns.

The Attitude Section of the Survey

The attitude section of the survey contained 40 statements about the teacher's attitudes towards mathematics and mathematics education. The teacher used one of five categoires (from Strongly Agree to Strongly Disagree) to respond to each statement. The statements were hypothesized to represent four randomly-merged subscales of 10 items each-- attitude toward self as learner of mathematics, attitude toward self as teacher of mathematics, attitude toward students as learners of mathematics, beliefs about what mathematics is.

The Procedure

The survey was sent to all teachers of mathematics, grades K-5/6, even those who taught only one class of mathematics, through a designated teacher in each building. The response sheets contained no identifying information other than that requested in the items and were returned to the designated teacher in each building, who then returned them to the Elementary Supervisor. The anonymity of the responder and the non-involvement of principals were intended to provide maximum opportunity for openness in responses. There were 259 respondents to the survey, representing a return rate of more than 95%.

The responses to the attitude section of the survey were subjected to factor analysis to determine the construct validity of the hypothesized subscales. If the hypothesized structure could not be substantiated, the factors would be used to define subscales appropriate to the structure of the items. A significant loading of a variable on a factor was defined to be |.30| or greater. Numerical values of responses to certain items were reversed so that all "favorable" responses would have the same value.

Results

The factor analysis was performed using the factor procedure in SAS (1979) with iterated principal factor analysis, followed by Varimax (orthogonal) or Promax (oblique) rotation. The Kaiser-Guttman criterion (i.e., minimum eigenvalue of 1.0) indicated 13 factors. However, an examination of the 13-factor solution showed many singleton and doubleton factors. The solution was rejected. Cattell's scree test (1978) indicated 5 or 6 factors. The hypothesized structure indicated 4 factors. The 4-, 5-, and 6-factor solutions were examined. The orthogonal and oblique solutions for a given number of factors differed only slightly. Both 6-factor solutions contained a doubleton factor and were rejected. The only difference between the 4- and 5-factor solutions was the split of the first factor in the 4-factor solution into two factors in the 5-factor solution. The 5-factor oblique (Promax) solution was accepted.

The 5-factor solution of the attitude items of the survey did not correspond exactly to the hypothesized subscales. Space does not permit the reproduction of the entire table of factor loadings. The items loading highest on each factor are given in Table 1 in descending order of their loadings.

Table 1
Items Loading on Each Factor

Factor 1:	Enjoyment of Teaching Mathematics
I 11	Math is one of my favorite subjects to teach.
I 18	I look forward to teaching math every day.
In 3	I would rather teach reading than math.
I 19	I find many math problems very interesting.
In15	Math is more difficult to teach than reading.
In 7	Math is so hard to understand that I do not like it as well as other subjects.
I 34	I think math is beautiful.
In16	There is so much hard work in math that it takes the fun out of it.
Factor 2:	Ease in Learning mathematics
In 2	I find many math problems too difficult to be enjoyable.
In22	Many sutdents get very nervous before math tests because such tests are very difficult.
In 4	No matter how hard I try, I find math difficult to understand.
I 12	I have usually found math courses easy.
I 30	Students would like math better if it were not so hard to understand.
In 5	Even the thought of taking a math course scares me.
In13	Math courses have usually been one of my weak spots.
I 29	Math is more of a game than it is hard work.
I 17	I understand math concepts easily
In32	Most students must work very hard to do well in math.
I 26	Most people can learn the math taught in elementary and secondary schools.
Factor 3:	What Mathematics Is
I 39	Math should be enjoyed for its own sake.
I 35	Most of my students can understand the math I teach.
I 23	Problem solving is an integral part of math.
I 28	Math is a good subject for mental discipline.
In36	Math class is not a place for a student to show creativity.
I 21	Arithmetic computation is only a small part of math.
In31	Students would like math better if it were not so hard to understand.
Factor 4:	Students as Learners of Mathematics
In37	Boys are usually better at math than girls.
I 8	I believe it is just as important for students to understand math as to be able to do computation.
In38	Most math is too concerned with ideas to be really useful.

In33 Many girls have math phobia
In27 To do well in math class, you have to be smarter than you have to be to do well in other subjects.
In24 Math can only be understood by a few.

Factor 5: How to Teach Mathematics
In14 I believe that math classes should be mostly practice work.
In 6 I feel that manipulatives are only appropriate in math classes in the early grades.
In25 Math is basically a collection of rules about numbers.
In20 I believe teachers should not try to prove math statements to children until at least high school.
In 9 When my daily class schedule is disrupted, I skip math class rather than other classes.

Items not loading significantly on any of the factors:
In 1 Problem solving cannot be taught until children have learned the basic arithmetic skills.
In10 I do not understand how some people can say that math makes a lot of sense to them.
I 40 Most students find math interesting.

Items for which the numerical values of the responses were reversed are indicated with an "In" preceding the number; other items have only an "I" preceding the number. Items In1, In10, and I40 did not load significantly on any of the factors.

Discussion

The factors of the attitude section on the survey are quite independent. They have been named as follows: Enjoyment of Teaching Math, Ease in Learning Mathematics, What Mathematics Is, Students as Learners of Mathematics, and How to Teach Mathematics. For further analyses, the items which load highest on each factor will be used to define a subscale of the attitude section of the survey. An arithmetic function of the responses to each attitude subscale will be defined to serve as measurements of the variables represented.

It is important to note that the correlation of factors 1 and 2 indicates that a teacher's perception of the ease or difficulty of learning mathematics and the teacher's enjoyment of teaching mathematics are related. Also, the items about the teacher's difficulty in learning mathematics, as well as the items about a teacher's perception of the student's difficulty in learning mathematics, load on the same factor. If either of the attitude subscales defined by these factors is significantly related to student achievement, that relationship would have consequences for the education and selection of mathematics teachers for elementary students.

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JOHN'S ARITHMETICAL ADDING AND SUBTRACTING SCHEMES

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The adding and subtracting schemes of John, a second grader, were characterized as arithmetical in a teaching experiment involving eight children in Spring of 1983. The arithmetical schemes were identified as one of four developmental levels of children's adding and subtracting schemes. John's schemes did not involve counting, rather, he operated directly on numbers and recalled number facts to solve all types of addition and subtraction tasks. His schemes included partitioning, recalling sums using doubles, adding to a decade, and recalling sums using place value. He understood part-whole relationship and subtraction as the inversion of addition. His addition and subtraction concepts were operative.

This paper characterizes the arithmetical adding and subtracting schemes of John, a second grader. Children create countable items, in order of sophistication, by counting with perceptual, figural, motor, verbal or abstract unit items (Steffe, von Glasersfeld, Richards & Cobb, 1983). John did not use counting at all to solve tasks even when he was hinted to do so, but he had constructed numbers so he was classified as a counter of abstract unit items together with four other children (Eshun, 1985a). The other four children found counting crucial in solving addition and subtraction tasks.

DEVELOPMENTAL LEVELS IN CHILDREN'S SCHEMES

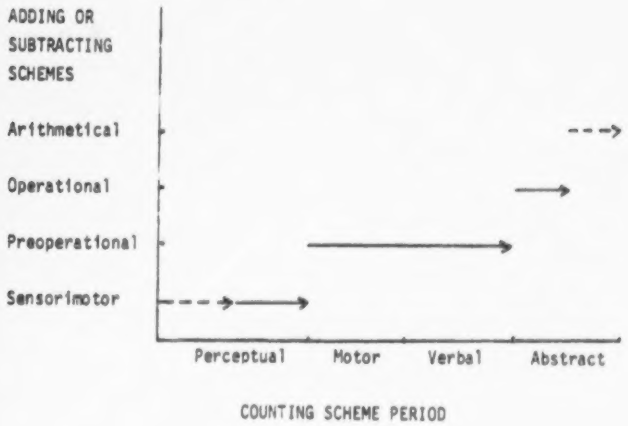
Piaget (1952) classified children in his third stage of development as having arithmetical understanding of addition and subtraction. While the behavior of the four other counters of abstract unit items revealed some of the characteristics of Piaget's (1952) third stage (cf. Eshun, 1985a). John's behavior was far more advanced in several other ways than the other children. For example, John had fully internalized his operations of addition and subtraction, and could partition independently composite units (addends) and recombine them with other addends. Also, he could decompose independently a composite unit into units of tens and units of ones. He used his understanding of part-whole relationship to solve several tasks including missing-addend, comparison, and equalizing tasks.

Figure 1 shows the revised developmental levels of Eshun's (1982) proposed experimental model of children's addition concepts. The four developmental levels, in order of sophistication, are sensorimotor, preoperational, operational and arithmetical schemes. Each child's schemes and concepts in the revised model were limited to only one developmental level. John was classified as using arithmetical adding and subtracting schemes whilst the four other children were classified as using operational schemes, one level below John's (cf. Eshun, 1985a, 1985b).

The revised model was based on the interpretation of the behavior of eight first and second graders involved in a teaching experiment in Spring of 1983. Individual teaching episodes were held with the eight children selected from among 25 others in a school in Clarke County, Georgia. Each child was taught from four to six times, and each episode was video recorded and lasted from 20 to 30 minutes. Addition and subtraction tasks were presented using a function machine and numerals on cards, ranging from "2" to "61".

JOHN'S ARITHMETICAL SCHEMES

John's ability to partition numbers spontaneously was the underpinning strategy for most of his schemes. Also, John showed flexibility in recalling number facts. He combined his partitioning strategy and the recalling of addition and subtraction facts to construct specific thinking strategies for solving tasks. John's schemes were synonymous with



Legend: --> Level for more advanced counters during this part of the period

→ Level for all counters during this part of the period.

Figure 1: Revised Developmental Levels of Children's Adding and Subtracting Schemes and Counting Scheme Periods.

Carpenter and Moser's (1984) derived facts and Houlihan and Ginsburg's (1981) indirect method. His concepts of addition and subtraction were, therefore, operative (Piaget, 1970) or numerical.

Recalling Sums Using Doubles Scheme

In this scheme John recalled an addition fact that was the double of one of the given addends. He then increased or decreased the partial sum to obtain the required sum. Consider the explanation to his solution to the missing-addend task "N(15) + N(__) --> N(31)."

> J: If 15 plus 15 is 30, you need one more to make 16 and it should be 31.

Adding to a Decade Scheme

In this scheme John interpreted an addition task as adding a number to the larger addend to yield the next decade. He then increased the new decade to get the smaller addend. The following illustrates how John used the scheme to explain his solution to the task "N(8) + N(7)."

> J: You have eight and take two from seven and you get 10, and that leaves five left. And you take five and add it to the 10 and is 15.

This is a more general scheme than a relating to ten scheme, because John used the scheme to solve tasks involving larger decades, for example, "N(34) - N(__) --> N(20)."

Recalling Sums Using Place Value Scheme

This scheme involved recalling the sum for the numbers in the tens and ones places separately and coordinating the two sums to form the appropriate number (result). For example, to solve the task "N(13) + N(15)," John answered, "Twenty eight" (immediately) and explained, "You've 13 and 15 and you take the five and three and add them up and put eight down. And you take the two ones you have left and add, and that gives you two." This scheme is consistent with the algorithmic procedure children are taught in school. But the scheme might be a construction by John, because it was not used by Jeff from the same second-grade classroom.

Inversion Strategy

John understood *subtraction as the inversion of addition*. This was evidenced by his inverting subtraction tasks into additive tasks before solving them. For example, John used the inversion strategy and the adding to a decade scheme to explain his solution to the missing-subtrahend task "N(34) - N(__) --> N(20)."

> J: Because you need 10 to make 30 and four more is 14, and that makes 34.

John's coordination of 10 and four to obtain 14, and 20 and 14 to get 34 also indicated his understanding of part-whole relationship.

DISCUSSION

John was the only one of the eight children to be classified as using arithmetical schemes. His behavior clearly showed arithmetical understanding of addition and subtraction (Piaget, 1952), and meaningful habituation of number combinations (Brownell, 1928). Steffe et al. (1983) identified Christopher, who used a recalling sums using doubles scheme to solve "7 + 5 = __" (p. 106), but he did not consistently use recalled number facts and thinking strategies like John.

There was a clear distinction between John's arithmetical schemes and the operational schemes of the other four counters of abstract unit items, because the latter found counting crucial and hardly used thinking strategies. But Carpenter and Moser (1984) identified 40 percent of the children in their three-year longitudinal study as using five or more derived facts and 80 percent of the children as using derived facts at some time. Thus, the question of what proportion of a child's schemes should involve recalled number facts in order to be characterized as using arithmetical schemes as John is significant and requires further research. However, the significant aspect of John's behavior was the flexibility in the use of his schemes.

Note: This paper is based on the author's doctoral dissertation at the University of Georgia under the direction of Leslie P. Steffe.

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A FUNNY THING HAPPENED ON THE WAY FROM THE PRESENTATION OF AN EVERYDAY PROBLEM TO ITS SOLUTION

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A mathematical problem, phrased as an everyday situation, may cause difficulties in the solving process, even for good students. A solution of such a problem may have only one stage - a direct one, or may have two stages: the first is to find a mathematical model of the problem, and the second is to solve the mathematical problem.

Analyzing responses of good students may show us the most essential difficulties in the solution of this kind of problem. In this study we have analyzed the answers of good ninth grade students for two problems phrased as an everyday situation. The main results are: Constructing a mathematical model seems to be a necessary stage for a solution, and the most common errors for this kind of problem are caused by basing the solution on arguments that appear correct, but are not suitable to the data and the situation described.

INTRODUCTION

"Indeed this is probably the most crucial and largely unsolved difficulty of mathematics teaching in schools - to enable pupils to identify the mathematical task - however simple - required in an everyday situation". (Eggleston, 1983).

The difficulty depends, of course, on the given problem. Analyzing students' answers in this subject may lead us to a better "attack" on the weak points and on the typical errors.

The student population we are dealing with are pupils in grade 9 who have participated in a mathematics correspondence course given by the Youth Activities Section at the Weizmann Institute of Science in Rehovot (Even & Kreimer, 1983). This course is meant for students who are motivated to occupy themselves with mathematics in their free time, so we can assume that they are above average ability in mathematics. The analysis of the responses is obviously important in the case of average children and for low achievers, but is also important for above average students. Although these students indeed make relatively fewer errors, and less effort is required to bring them to a correct solution, the analysis of their responses can teach us much about their mental processes. The errors of good students can teach us the most essential difficulties in any kind of problem, and their analysis can help us to choose and word the problems according to our aims, - in preparing textbooks, worksheets and enrichment sheets, or in asking questions in classroom lessons.

In this paper we shall present two different problems phrased as everyday situations, and we shall analyze them according to similarities and differences. We shall try to classify the students' answers according to the typical errors in the method of reaching a solution, and attempt to identify their causes according to the level of the student and the kind of problems given.

PROBLEM - MATHEMATICAL MODEL - SOLUTION

The two problems are:

The problem of the bakeries -

A baker who owns 2 bakeries decided to build a flour storeroom to be used for both. Where should the storeroom be built so that transporting the sacks of flour from the storeroom daily will be cheapest? (Suppose that the daily requirement of flour for each bakery is constant).

The problem of the greeting cards -

In honor of Israel's Independence Day, each city sent a greeting card to its nearest neighboring city. Supposing that the distances between cities are *different, prove that each city got no more than 5 such greeting cards for Independence Day.*

Both problems are phrased as problems in everyday reality, where the main difficulty in solving them is the stage of rewording them as mathematics problems:

The problem of the bakeries -

Let s be the distance from bakery A to bakery B.

x - *the distance from the storeroom to bakery A.*

c - *the cost of transporting a unit of flour a unit of distance*

p - *the daily consumption of flour at bakery A.*

q - *the daily consumption of flour at bakery B.*

y - *the daily cost of transportation.*

Now a suitable mathematical model for the original problem can be constructed:

What is the minimal value of the function $y = pxc + q(s - x)c$, *when* s, c, p, q *are constants?*

The problem of the greeting cards -

Given a finite number of points in the plane at different distances from each other. Prove that there is no polygon (whose vertices are the above points) whose number of sides is greater than 5 and fulfils the condition: the distance from a point inside the polygon to any of its vertices, is smaller than the length of each of the two sides that meet in that vertix.

After overcoming the first stage of constructing a mathematical model we reach the second stage - the solution.

The problem of the bakeries -

a) If $p = q$ *the function is*

$$\begin{aligned} y &= pxc + p(s - x)c \\ &= pxc + psc - pxc \\ &= psc \end{aligned}$$

That is, when the daily consumption of the two bakeries is equal, the function is constant. Therefore, the storeroom may be located anywhere on the line joining the two bakeries.

b) If $p \neq q$ then the function is

$$\begin{aligned} y &= pxc + q(s - x)c \\ &= pxc + qsc - qxc \\ &= xc(p - q) + qsc \end{aligned}$$

i) If $p > q$ then one gets the minimum of the function when x is minimum, that is $x = 0$.

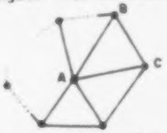
ii) If $q > p$ one gets the minimum when x is maximum, that is $x = s$.

In both cases we found that if the daily consumptions of the two bakeries are not the same, the storeroom should be placed in the bakery whose daily consumption is biggest.

The problem of the greeting cards -

Suppose there is a polygon with more than five sides, and a point A in it. Let us draw lines from A to each vertix. Since there are at least six vertices there will be at least six angles having A as their vertix.

The sum of the angles is 360°, so at least one of the angles will be smaller than or equal to 60°. Suppose it is the angle BAC.



If $\sphericalangle BAC < 60^\circ$ then $BC < AC$ or $BC < AB$. If that is so, a polygon of more than five sides cannot satisfy the desired condition: the distance from the point inside the polygon to each vertix of the polygon is less than the length of each of the two sides of each vertix.

STUDENTS' RESPONSES

We can divide the student-solvers into three main groups:

a) those who do not try to go over to a mathematical model.

b) those who manage to get to a partial mathematical model.

c) those who succeed in getting to a completely mathematical model.

An examination of the students' responses shows that almost all of them tried to find a mathematical rephrasing of the problems; that is to say, they felt the need to convert the problem to a mathematical one.

The problem of the bakeries, because of its lack of data and because its wording is open and invites original suggestions and does not compel the student to prove any given statement, encouraged the students to provide practical solutions. Here are two examples for such suggestions:
"The baker must locate the storeroom at one of the bakeries, otherwise he has to buy more land".
"If there is a difference in height between the bakeries, he has to build the storeroom at the highest bakery in order to save gasoline." Such errors were typical of students that did not make any attempt to go over to a completely mathematical model.
But even a student who solved the problem correctly with the help of a minimum of function, ended his solution as follows:
"Therefore the storeroom must be built in Bakery A (or as close as possible to it) so that the transportation will be cheapest".
Some of those students who attempted to solve the problem by way of a solution from everyday life, without going over to a mathematical phrasing, ignored the problem of the site of the storeroom and offered alternative methods of decreasing expenditures, such as paying less to the drivers, building a cheaper storeroom, etc.

Such phenomena did not appear in the solutions to the problem of the greeting cards. Evidently this was so, because from its wording it was clear that a mathematical proof was required.
The typical errors made by those who tried to solve the above problems were caused by their basing themselves on arguments that perhaps appeared correct, but were not suitable to the data given in the problem and the situation described. Here is an example for such a "solution" of the problem of the bakeries:
"If the consumption of flour by the two bakeries is the same, then the storeroom must be located in the middle between them. But if the consumption is not

the same we have to take it into consideration. F r example, suppose the daily consumption of bakery A is 10 tons, and the daily onsumption of bakery B is 5 tons. The relation between them is 10:5 that m ns 2:1. Therefore, the storeroom should be built $\frac{1}{3}$ of the distance betwe the bakeries, closer to the one that uses the most flour." The pupil bas it on an argument that seems correct: the relation between the distance from the bakeries to the storeroom has to be opposite to the relation betw n the consumptions. But this argument is unsuitable for the given situati This phenomenon originated in the fact that there as no transfer to a completely mathematical model.

In conclusion, we would like to point out that the construction of a mathematical model that suits the problem phrased as an eve day situation was a necessary condition for solving the problem. Not one the students succeeded in reaching a correct solution without first rephras the problem in mathematical terms, (partly or wholly). Therefore it is n essary to work on this stage with the pupils and to make them conscious it.

The most common mistake in solving problems of th above kind was made by students basing themselves on mathematical argume that did not suit the situation described. This happened because there as no transfer to a complete mathematical model. Another common mistake was t inclination to a practical solution with, or even without, a connection to w t the problem demanded. Such a mistake was typical of pupils who made no tempt to rephrase the problem in mathematical terms.

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OPERATING THE UNKNOWN AND MODELS OF TEACHING
(A Clinical Study Among Children 12-13 with High Proficiency in Pre-Algebra)

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ABSTRACT.- Operating the unknown is one of the first properly algebraic actions. As reported in a former paper [2] extending arithmetical operations to new objects (e.g., the unknown) is not a spontaneous process in children, and requires special attention in the field of teaching. In this paper, we are reporting the most relevant results from the research work "Operating the Unknown" which refers to the constant, as well as to the variable aspects that were observed when using various models for operating the unknown in the resolution of linear equations having the form $Ax \pm B = Cx \pm D$, where, A, B, C and D are given positive integers and $D \geqslant 0$. Observations were performed through clinical interviews to children 12-13 who had had no previous instruction in algebra, but had showed a high proficiency in pre-algebra. By way of conclusion, our study shows that all teaching strategies should contemplate a dialectical process between the most concrete meanings and the operational syntax, because both aspects are always present (due to the students' anticipatory mechanisms), even if they have not been taken into account at the time of devising the didactic situations (as it occurs in the usual teaching strategies).

INTRODUCTION.- It has always been thought that more concrete models possess the virtue of providing more stable meanings to new concepts, whereas more syntactic models have a tendency to introduce a certain senselessness to everything new that is taught. It is also commonly believed that the latter aspect is the one that contributes most strongly to the lack of interest on the part of children towards the study of mathematics (at the affective level), and to the tendency to commit the typical and spontaneous errors (cognoscitive level) when operative abilities required by algebra come into play. The results of this paper, however, show that even models where meanings are taken from a more concrete language, will pose, when used as teaching strategies in the resolution of linear equations, the same problems; naturally, these problems will have particular manifestations depending on the model under consideration.

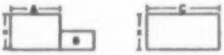
GENERAL FRAMEWORK AND SPECIFIC OBJECTIVES OF THIS WORK.- "Operating the Unknown" is a part of the research program "The Acquisition of Algebraic Language", which has been developed since 1980 at the Sección Matemática Educativa and the Centro Escolar Hermanos Revueltas, in Mexico City. Probing into the difficulties involved both in the syntactic handling of algebra, and the utilization of algebra to solve problems, in relation with didactic phenomena that appear during the child's transit from arithmetical to algebraic thought, are among objectives of the wider research. This study, "Operating the Unknown", has the following specific aims: 1) to observe the spontaneous responses of children that are faced, for the first time with "non-arithmetical" equations (i.e., those where resolution demands operating the unknown, as, for instance, in some linear equation with more than one occurrence of the unknown) and 2) to observe how children develop by themselves the use of some "concrete" model for operating the unknown in the solving process for these new equations. Results referring to objective 1) have been reported in a previous article [2] ; in the present paper, we describe results concerning objective 2), where two different "concrete" models are being tested in order to teach how to operate the unknown.

METHODOLOGY - Detection and observation of transition phenomena were performed through clinical interviews to three successive classes of children 12-13, within a controlled teaching system and at the time when children already know how to solve "arithmetical" equations (i.e., those whose solutions does not require operating the unknown) and related problems, and are faced for the first time with the resolution of "non-arithmetical" equations. During the interview, and after observing their spontaneous approaches to the new equations, children with medium and high pre-algebraic proficiency levels are given instruction elements for the operation of the unknown, by making use of two "concrete" models in the resolution of such equations (results presented here refer only to observations made among children with high pre-algebraic proficiency).

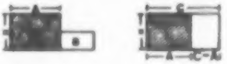
A SYNOPTIC DESCRIPTION OF THE TWO MODELS USED.

I. **The geometric model.**- Proposed equation: $Ax + B = Cx$, where A, B and C are given positive integers, and $C > A$, in this case.

1. Reproducing the model (translating the equation into the model).



2. Comparing areas.



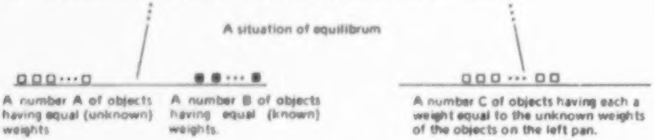
3. Preparing the simplified equation: $(C - A)x = B$.

4. Solving the simplified equation.

5. Verifying the answer.

II. **The balance model.**- Proposed equation: $Ax + B = Cx$, where A, B, and C are given positive integers, and $C > A$, in this case.

1. Reproducing the model (translating the equation into the model).


A situation of equilibrum
A number A of objects having equal (unknown) weights
A number B of objects having equal (known) weights.
A number C of objects having each a weight equal to the unknown weights of the objects on the left pan.

2. Iteratively reducing the number of objects with a known weight, although maintaining a balance, until all such objects have been removed from one of the pans.

B objects
■■···■■
C − A objects
□□···□

3. Preparing the simplified equation $(C - A)x = B$.

4. Solving the simplified equation.

5. Verifying the answer.

In both models, children with a high pre-algebra proficiency level are given only the first elements of the model (translation phase 1), and are left to develop subsequent stages by themsselves or with the least possible help from their interviewer. Once they have mastered the use of a model for one mode of equation ($Ax + B = Cx$), more and more complex modes ($Ax + B = Cx + D$; $Ax - B = Cx - D$; etc.) are proposed to them, in order to observe how they transfer the use of the model to these modes, as well as the abstraction processes of repetitive operations on the model.

RESULTS.- a) Spontaneous development of the use of the model to operate the unknown does not show a uniform pattern, not even among children having the same level of pre-algebra proficiency; such a development is strongly linked to tendencies of a general nature in the subject and which range from the syntactic or operational perspective on one end of the spectrum, to the semantic perspective on the other. In fact, extreme cases were detected, presenting remarkably dissimilar develpment paths in the use of the same model; in one first case, this development was carried out with a permanence in the model context, even in modes of equations presenting a highly complex model structure. In the other contrasting case, that is, when dealing with an operational tendency, there is a constant search for the syntactic elements present in the model's actions, that are repeated for each equation and for each equation mode; this search provokes a quick aloofness from the model's semantics, in order to inscribe these actions within a more abstract language, by creating personal graphs (signs or symbols) that belong neither to the model nor to algebra, but to an intermediate level preceding the algebraic operational level.

b) There are obstructors to abstraction of the model's operations towards a syntactic-algebraic level, that depend neither on the particular model being used, nor on subjects' tendencies such as those mentioned under a) above; they depend on the emphasis placed in the component of the model structure that permits relying on previous knowledge, and operations already mastered by the subject, in order to introduce the new objects, concepts and operations; this process of reducing the situation to what is already known, carries with it the risk of hiding such difficulties as arise when trying to operate with new objects and making new concepts to intervene. Thus, in the processes of abbreviating and automating the actions in the two models used here, a tendency is seen towards hiding the actual operation of the unknown. In the geometric model abbreviation leads to a blurring of areas involving the unknown; in fact, the linear dimension lost is the one that represents the unknown, and operations are performed only between the equation's "data": the unknown stops playing a role. In the balance, due to the discrete representation of x's coefficients, as well as of constant terms, the same type of operations can be performed with both, i.e., operations between numbers of objects with known weight and numbers of objects with unknown weight. Such an automation (in both models) will later on lead to the typical mistakes in algebraic syntax, such as effectively adding (or substracting) terms possessing different degrees; even subjects with a great operative tendency commit this kind of mistakes, due to their use of personal graphs, which are also generated in a process of automating the actions.

The following are some of the aspects that vary from one model to the other:

c) There are specific ways (depending on the model) of translating the equation's elements into the model, that represent an obstruction to the use of the latter; in the geometric model, such an obstructor consists in breaking up the rectangular area represented by the constant term (B, in $Ax + B = Cx$) into linear dimensions; this leads to the application of the "coupling of linear dimensions" method for resolution purposes (i.e., to find b and h, such that $b \times h = B$ and $b = C - A$ or $h = C - A$), which is not applicable if B cannot be divided by $C - A$. In the balance, trying to assign weight to the objects in the pan can cause a confusion in the development of the model's initial "natural" strategy, to wit, the iterated cancelling of weights.

d) Some transfers in the use of the model, from one mode of equation to the other, are more "natural" in one model than in the other. Passing from the $Ax + B = Cx$ mode to the $Ax + B = Cx + D$ mode is more natural in the balance, since the action of iterative cancellation is essentially the same in both modes; besides, in this model the simplified equation stands stated in the model itself and can be solved without having to translate it into the graphs of algebra. In the geometric model, on the other hand, it is necessary to realize that in order to state the simplified equation it should suffice to superimpose corresponding areas to terms of degree one, without effecting any action on those corresponding to constant terms, i.e., transference in this case is not trivial. However, passing to modes such as $Ax - B = Cx$ or $Ax - B = Cx + D$, requires interpreting in the model the "negative" constant terms, which have no "concrete" representation in the balance (unless one resorts to representations of mental actions, as, for instance reestablishing an equilibrium), whereas in the context of areas, such terms can be interpreted as "concrete" actions consisting in the removal of areas equivalent to the absolute values of the terms in question, without thus making violence to the model's semantics.

CONCLUSIONS.- With respect to the constants observed in these two models (results a) and b)) the conclusion can be reached (and this applies also the observations) that only when it is possible both to attribute meanings to the new objects, situations, and operations, as well as to provide these with new senses within the over-all perspective of the solving process, will teaching strategies based on these models permit to reach the desired levels of abstraction in the operations. In order to confer these new senses to objects and operations, the following are required:

i) a contrasting process of the various concrete situations that arise from the use of the model; ii) a certain level of awareness that such differences exist; and iii) a good syntactic level that avoids anchoring at a more "concrete" language level, at the time of momentary loss of previous abilities or when losing the capacity to operate with more "concrete" objects, once this capacity had already been developed. (See [3] Filloy, E./Rojano, T., 1985).

As to variations from one model to the other (results c) and d)), some of these seem to indicate that one of the models, for certain modes of equations, favors more than the other the transfer to a syntactic approach (due to the difficulties involved in the evocation of a model in some modes of equations, for instance), whereas the other favors progress — because of its iterative application — in the resolution of several modes of equations; these variations, however, as already mentioned in a) above, are modulated by the type of tendency shown by the subject. On the other hand, both models share limitations, such as the fact that they have no way of representing, within their respective contexts, equations with a negative solution; for the treatment of such equations using the model-structure as a starting point, it is necessary to have previously undergone a process of operational abstraction, that such operations have been extracted from a syntactic level, and that they are applied, at this level, to the solution of equations; in other words, new operations must have been given meaning through processes i), ii) and iii) mentioned above. The only paradigmatic model into which it seems to be feasible to translate every mode of linear equation is algebra itself, with or without the usual graphs and codes.

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A FRAMEWORK FOR PROBLEM SOLVING

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Four junior high school students were interviewed to investigate the role of mathematical beliefs in problem solving. Mathematical beliefs were found to influence what these students did when they solved problems. Problem solving was conceptualized as being part of a larger context or framework with many components.

The purpose of this study was to investigate the role of students' mathematical beliefs in explaining their mathematical problem solving processes. The subjects were four junior high school students participating in Purdue University's STAR program, a two-week intensive summer program for the mathematically and verbally talented. These subjects were enrolled in the beginning section of a Math Problem Solving with Computers class. These four students, Sara, Dan, Cindy, and Mark were individually interviewed throughout the program. The interview sessions were audiotaped. In the interviews the students were questioned about their classroom experiences in mathematics and were encouraged to discuss their beliefs about mathematics. However, most of the interview time was spent with the student using the "think-aloud" technique to solve problems.

The premise on which this study was conceived, designed, and implemented was that problem solving does not depend only on cognitive factors. Other elements, such as mathematical beliefs, are necessary to explain what an individual does when he solves a problem. This is not a completely new idea in mathematics education problem-solving research (see, for example, Silver, 1982; Schoenfeld, 1983; Confrey, 1984).

MATHEMATICAL BELIEFS

At the time that this study was designed and carried out, the focus was entirely on the role of mathematical beliefs in explaining problem-solving processes. After a preliminary analysis of the interview data, five general categories of mathematical beliefs were distinguished. All of these beliefs were found to influence problem-solving behavior.

Beliefs about one's ability to do mathematics. All of the interview students confidently tackled problems which appeared routine to them. Clearly they believed they would be able to solve such problems. This may account for their avoidance on some problems of what could be called the understanding/exploring mode of problem solving. They may have been so confident of their ability that they did not feel a need to explore, but would instead immediately select a plan and carry it out.

Beliefs about mathematics as a discipline. Examples of beliefs about mathematics as a discipline are Sara's statement that "math has a definite answer" or Cindy's claim that "math is figuring." For the interview students, math was what they all called "the basics" (addition, subtraction, multiplication, and division) plus a collection of routine problems with "definite answers" which could be obtained relatively quickly and without too much effort through the application of known arithmetic or algebraic algorithms. The interview students' beliefs about mathematics as a discipline were related to their problem-solving behavior. They had a tendency to quickly bail out on problems which did not appear routine. This may have been because they all, like Sara, perceived such problems as "extra-credit" problems--not really mathematics.

Beliefs about where mathematical knowledge comes form. For most of the students in the beginning section of the Math Problem Solving with Computers class, mathematical knowledge, at least in the case of programming, came from The Teacher. (All of these students agreed with a pretest questionnaire statement "computer programming is a kind of mathematics.") Their purpose for doing this type of mathematics was to get right answers. The teacher, at least at the beginning of the STAR program, was the only one who could tell you if your answer was right. Later, a few of the students were willing to ask other students if their programs were right, or even to test the "rightness" by running the programs on a computer.

Beliefs about solving mathematics problems. Further analysis of the interview data revealed at least five types of beliefs that these students held about solving math problems: beliefs about what counts as a math problem, beliefs about what strategies are appropriate, beliefs about when a problem is solved and what constitutes an acceptable answer, beliefs about how long it should take to solve a problem, and beliefs about what to do when one gets stuck while trying to solve a problem. Space does not permit discussion of each of these types of mathematical beliefs. The interview students did seem to share the belief of Schoenfeld's (1985) college students that "mathematics problems are always solved in less than 10 minutes if they are solved at all." This belief has what Schoenfeld calls a corollary: "give up after 10 minutes." This corollary also held in the case of the STAR interview students.

Beliefs about how mathematics should be taught and learned. Again, a more detailed analysis of the data suggested that there were several components of this type of mathematical belief. Some of these components included beliefs about the role of the mathematics teacher and the mathematics student. These beliefs were so stereotyped that they seemed to be part of a set of mathematics myths, rather like the civic myths (such as the myth of the good citizen) discussed by some political scientists (Wilker & Milbrath, 1972). After reviewing some of the political science and math anxiety literature on myths and some studies of teachers' and students' beliefs about mathematics, I developed several categories of math myths which also seemed useful in explaining the interview data. These include the Myth of the Good Math Student and the Myth of the Good Math Teacher. The Good Math Teacher transfers information to the student through explanations and demonstrates procedures and methods. The Good Math Student follows directions, listens to the teacher, does all his homework, and gets good scores on tests.

DEVELOPMENT OF THE FRAMEWORK FOR PROBLEM SOLVING

After a preliminary analysis of the interview data it became apparent that beliefs alone could not adequately explain what these students did when they solved problems. Mathematical problem solving, for these students at least, was conceptualized as being part of a larger framework. Components of this framework include the individual's prior experiences in mathematics, his mathematical knowledge, and his mathematical beliefs, needs, and motivations. Mathematics myths (popular public beliefs) are also part of the framework, since they influence the individual's view of mathematics. Some of these framework components are discussed briefly below.

Needs, motivations, myths, and change. Maslow's (1970) taxonomy of basic needs was used to classify the "mathematical needs" of the four interview students as inferred from the interview data. Many of their mathematical needs appeared to fall into the category of safety needs, which are described by Maslow as involving the needs for the familiar, for security, stability, protection, structure, order, and freedom from anxiety and chaos. Safety needs are one of the lowest orders of needs in Maslow's hierarchy. Some of the interview students' mathematical needs which fell into this category included: the need to know if a problem is "possible" (has an answer), the need to know that your answer is right, the need to have the teacher tell you exactly what to do, and the need to know and remember all mathematical facts presented in class. According to Maslow, safety needs are particularly strong in children in our society because they tend to grow up in a threatening, non-loving family environment.

Given the mathematical needs of the interview students, it is quite reasonable that their motivation in mathematics tended to be what Nicholls (1983) calls extrinsic motivation and ego involvement as opposed to task involvement. The extrinsically-motivated or the ego-involved child views learning as a means to an end—for example, an ego-involved child wants to avoid looking stupid. The task-involved child, on the other hand, values learning and understanding for their own sake, and what he can get cognitively out of working on a task. The task-involved child seems to be operating at a higher level of need than the extrinsically-motivated or ego-involved child. If Maslow's theories are applicable here, it is completely unreasonable to expect children—gifted or not—to exhibit many higher order needs (such as the need for self-actualization or the cognitive or aesthetic needs) until their lower order needs have been satisfied. It is also easy to see why the math myths described earlier are so tenacious. The main function of myths is to make the world seem like a safer place. Since the child is supposed to have powerful safety needs, is it reasonable to expect that junior high school students, being children, are capable of experiencing many higher order needs? Is it reasonable to expect society's math myths to disappear and students' beliefs about mathematics to change overnight? Maslow puts the blame for children's overwhelming needs for safety on the family. Holt (1982) would put the blame not just on the family but on the schools.

Where do we go from here? Children's mathematical beliefs do seem to influence what they do when they solve problems. It seems unlikely that students can learn to become better problem solvers unless they learn to change their mathematical beliefs. But beliefs appear to be only part of a larger framework for problems solving and thus probably cannot change without changes in the rest of the framework.

One issue that needs to be addressed in future research is the applicability of the framework to the problem-solving behavior of other age and ability-level students. Informal observation of five sections of a college mathematics class for preservice elementary teachers suggests that the framework also fits what these students did when they solved problems.

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THE GROWTH OF SIMILARITY CONCEPTS OVER THE MIDDLE GRADES (6, 7, 8)

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Understanding similarity is essential in the development of children's geometrical understanding and of their ability to reason proportionally. The purpose of this study is to determine the extent to which sixth, seventh, and eighth graders exhibit an understanding of concepts of similarity. Six classes were taught an instructional unit on similarity, and fifty average-ability students were selected for pre and post interviews.

Student strategies on four rectangle similarity tasks presented in the interview showed little consistency within subject or within task. For each of the four tasks, the level of performance was strongly influenced by the numbers involved. As a result of instruction, 90 percent of the interviewed students employed same or higher level strategies as compared to their first interview: a considerable decrease in the number of students that operated on a visual level or employed an additive strategy could be observed.

INTRODUCTION

In the Proceedings of the Sixth Annual Meeting of PME-NA, October, 1984, the authors reported on a pilot study entitled "The Growth of Similarity Concepts at the Sixth Grade Level." The present study grew out of that pilot study.

Understanding similarity is an essential stage in the development of children's geometrical understanding of their environment. A firm grasp of the concept of similarity may also enhance children's development of porportional reasoning. Inhelder and Piaget (1958), consider proportional reasoning as one of the six abilities that characterize the formal-operational thinker. Even though the ability to use proportional reasoning is widely required in everyday life, there is a great deal of evidence that children have difficulty using such reasoning effectively (Karplus & Karplus, 1972; Pagni, 1983).

The present study has three major concerns--finding growth patterns in children's development of the concept of similarity over grades six, seven and eight; describing the effects of a specific instructional intervention over these three grade levels; and determining whether instruction in similarity has any effect on children's more general ability to reason proportionally.

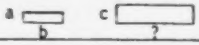
METHODOLOGY

The instructional intervention was the Similarity Unit developed by the Michigan State University Middle Grades Mathematics Project. The unit consists of nine activities designed to help students explore, in a concrete way, the concept of similarity and its applications. The unit was taught in six classes and took from ten to fifteen days of instruction.

Two classes at each grade level; six, seven and eight, comprise the sample for the study. The seventh and eighth grade classes had the same instructor. The sixth grade classes had different instructors. All three teachers had taught the instructional unit previously and were confident with the material. The students in all six classes were given two paper and pencil tests pre- and post-instruction. These tests were the Middle Grades Mathematics Project (MGMP) Similarity Test and a Ratio and Proportions Test which included the Mr. Tall/Mr. Short problem from a study by Karplus and Karplus (1972), and selected items from the Concepts in Secondary Mathematics and Science (CSMS) Ratio and Porpertion Test (Hart, 1982). In order to avoid a large student variability, eight to twelve students were selected from the middle range of test scores in each class. The fifty selected students were interviewed pre-post-instruction. These interviews took from 30 to 60 minutes each. The tasks to be performed were presented in a uniform way to each student. The students were asked after each response to explain their reasoning in detail.

The students were presented with four different kinds of tasks related to determining similarity of rectangles. Each of these tasks was varied along a numerical scale designed to test the student's facility with handling proportions of increasing numerical difficulty. Table 1 shows the four tasks and the four ways in which the numbers in the proportions were varied.

Table 1. Interview Questions

NUMERICAL TYPE a by b c by d	1	2	3	4
TASK	$a \mid b$ $a \mid c$	$a \nmid b$ $a \mid c$	$a \mid b$ $a \nmid c$	$a \nmid b$ $a \nmid c$
1. Decide whether two drawn rectangles are similar or not.	3 by 6 and 9 by 18	2 by 3 and 8 by 12	3 by 9 and 4 by 12	6 by 8 and 9 by 12
2. Decide whether two cut-out rectangles are similar or not.	2 by 4 and 6 by 12	2 by 3 and 6 by 9	2 by 6 and 3 by 9	4 by 10 and 6 by 15
3. Given the lengths of three sides of two similar rectangles, find the fourth side. 	2 by 6 and 6 by ?	2 by 5 and 6 by ?	4 by 12 and 7 by ?	6 by 10 and 9 by ?
4. Cut a strip to make a rectangle similar to a given one.	2 by 4 and 6 by ?	2 by 3 and 8 by ?	2 by 6 and 5 by ?	4 by 6 and 6 by ?

MAIN RESULTS

Student performance on the comparison of pairs of drawn rectangles (Task 1) and on the completion of a strip to a rectangle similar to a given one (Task 4) will be analyzed in more detail. Task 1 was chosen because it has a lower potential to attract visual answers as compared to the comparison of cut-out rectangles. The latter was given first in the interview (serving accordingly as a "warm-up" activity) and also allowed for manipulations (i.e., nesting the two rectangles to a common corner and visually considering the diagonals or the part "left over" in the larger rectangle).

Task 4 is mathematically equivalent to finding the missing measure in a pair of similar rectangles (Task 3). Task 3, however, is a frequently used activity and has the potential of attracting mechanical answers (i.e., an algebraic solution of the proportions). Task 4 was not used during instruction and requires a better understanding of the proportionality principle involved.

Level of success. Figure 1 presents the percent of students that answered correctly on Tasks 1 and 4 at each grade level, before and after instruction. The graphs clearly indicate the positive influence of the instruction. A strong influence of the numerical type may also be observed: for all tasks and grade levels, there is a considerable gap (20-45 percent) between level of performance with numbers that are divisible across rectangles (Types 1 and 2) and between cases in which such comparisons do not render whole numbers (Types 3 and 4). Figure 1 also indicates that after instruction, an average mastery level of above 80 percent has been achieved for the first two numerical types but not for the others.

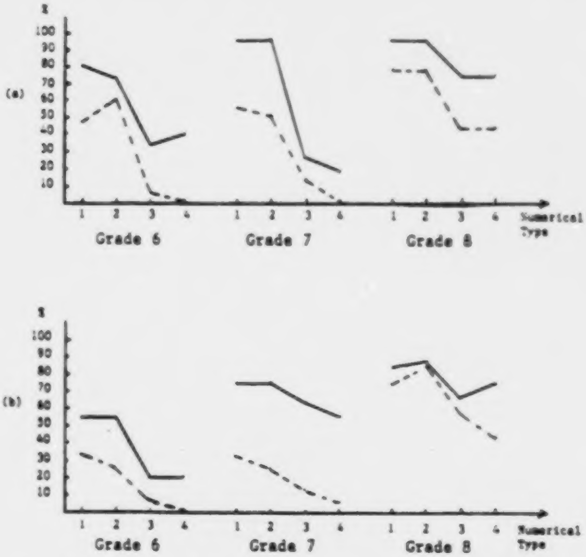


Figure 1. Level of success for Task 1(a), and for Task 4(b).

The average levels of performance on each of the four tasks did not allow a hierarchical ordering of tasks by degree of difficulty.

Analysis of Strategies. As stated in an earlier report Friedlander et al. (1984) student strategies in similarity tasks followed roughly the classification of responses for proportionality tasks indicated in Karplus and Karplus (1972). The following categories of responses were found in this study:

1. Visualization - using intuition without considering the lengths of the sides.
2. Addition - considering the difference rather than the ratio of the numbers.
3. Multiplication and Adjustments - multiplying to enlarge, and "adjusting" by subtraction or addition (e.g., in the proportion 2:6::5:x, x = 13 because 5 = 2·2 + 1, and thus 2·6 + 1 = 13).
4. Whole Multiplication - "fitting in" the sides of the small figure a whole (but not necessarily the same) number of times into the sides of the bigger figure. This kind of reasoning leads characteristically to the conclusion that if the scale factor is not an integer, the figures are not similar.
5. Proportional Reasoning - setting up two ratios and a correct or incorrect consideration of their equivalence, or more frequently, considering the scale factor by which the small rectangle is enlarged.

Student responses on Task 4 (Completion of Rectangle) for Numerical Types 3 and 4 were chosen for a more detailed analysis. As stated before, Task 4 requires a higher degree of transfer. Moreover, numerical Types 3 and 4 in this task were found to clearly distinguish between "whole multipliers" (i.e., children who give the right answer just by considering number divisibility) and "real" proportional reasoners: The "whole multipliers" would cut the strip at any multiple of the length measure of the given rectangle, without measuring or disregarding the measures of the given widths, whereas the proportional reasoners considered in most cases the fractional ratio of the two given widths.

Table 2 presents the distribution of strategies employed at different grade levels for this task. A comparison of pre- and post-instructional performance indicates a remarkable decrease in the number of students that operated at a visual level. It should also be noticed that before instruction, most sixth and seventh graders used the visual and the whole multiplicative strategies, whereas the eighth graders relied less on visualization and tended to employ the "more advanced" additive strategy.

Table 2. Distribution of strategies (in percent) employed in Task 4 for Numerical Types 3 and 4.

Strategy No.	Grade 6		Grade 7		Grade 8	
	Pre	Post	Pre	Post	Pre	Post
1	40	7	41	6	3	-
2	13	13	12	12	26	16
3	-	10	3	3	3	3
4	44	50	34	19	18	10
5	3	20	9	59	50	71

A Hierarchy of Strategies. Following the suggestions in Karplus and Karplus (1972), the strategies employed in the rectangle similarity tasks may also be hierarchically ordered. The list of strategies presented above is already arranged according to a developmental approach: the lowest strategy would be the intuitive visualization (1), followed by the additive scheme (2), a combination of addition and multiplication (3), and then an incorrect, or a correct scaling scheme (4 and 5 respectively). Karplus and Karplus tie their classification to Piaget by considering visualization, addition, and scaling as indicators of a subject's being respectively at an intuitive, preoperational, or operational level.

Table 3 presents a summary of the distribution of strategies employed in Tasks 1 and 4 for Numerical Types 3 and 4 by the whole sample before, and after instruction. In the two matrices, the numbers located on the main diagonal indicate no change of strategy between the two interviews. The upper, and the lower halves with respect to the diagonal mark students that employed more advanced, or respectively lower strategies in the post interview as compared to the strategy employed in the same task in the first interview. The results for both Tasks 1 and 4 indicate that about 90 percent of the students were either stable or advanced (with an almost equal division of

Table 3. Pre/post student progress on Task 1(a) and on Task 4(b) for Numerical Types 3 and 4.

PRE \ POST	1	2	3	4	5
1	3			11	12
2			1	1	4
3					1
4	3	4		27	14
5	1	1		2	15

(a)

PRE \ POST	1	2	3	4	5
1	4	3	1	6	12
2		5	1	2	[illegible]
3					[illegible]
4		6		16	7
5			2	1	22

(b)

of 45 percent for each of the two categories) and only about 10 percent employed in the post-interview lower level strategies than they did before.

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The Influence of Context Familiarity on Solving Word Problems in Mathematics at the High School Level

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Abstract

Since a famous study by Brownell & Stretch (1931), relatively few investigations have been conducted about the influence of context familiarity on solving word problems with a context external to mathematics. Results of a study conducted on that theme in Morocco, during 1983-84, will be presented here.

In order to study the influence of context familiarity, the first author prepared 16 word problems, that is 4 initial problems (on elementary number theory or algebra) each one couched in four analogous versions differing by a more or less familiar context. He used those sixteen problems to administer a test to 256 high school students from Morocco, and to further conduct individual interviews with 32 of them. Results obtained show that the degree of familiarity of the context -- familiarity being defined here in terms of the students -- has an unquestionable, appreciable influence on both the performance and the comprehension in solving mathematical word problems.

STUDYING CHILDREN'S USE OF HEURISTIC PROCESSES FOR MATHEMATICAL PROBLEM SOLVING THROUGH STRUCTURED CLINICAL INTERVIEWS

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This paper describes the creation of a script for conducting structured clinical interviews with academically talented children in grades 4-6, in order to study their use of the heuristic process "think of a simpler problem" (TSP). The children are guided through a complex plan to solve the problem, "What is the remainder when two to the 50th power is divided by three?" and their competencies in various subprocesses of TSP are observed. The prototype script is also intended as a model for the study of other heuristic processes.

BACKGROUND

In earlier work, Goldin and Germain (1983) and Goldin (1984, 1985) proposed that heuristic processes could be analyzed into subprocesses with respect to four main categories: (1) advance planning reasons the solver may have for using the process, (2) ways of applying the process, (3) domains to which the process may be applied, and (4) prescriptive criteria suggesting that the process should be applied. To be able to study the psychological structure of heuristic processes as they develop, it is important to give operational meaning to such an analysis--i.e., to measure children's competencies in the use of particular subprocesses in as much detail and with as much reliability as possible. The present paper describes the creation of a script for the use of clinicians during structured interviews, designed to observe academically talented children's use of various subprocesses of the complex heuristic process "think of a simpler problem" (TSP) discussed in the above-cited papers. The prototype script is also intended to serve as a model for the study of other heuristic processes that have been similarly analyzed, such as "special cases" and "trial and error."

DESIGN OF THE SCRIPT

Clinical tape-recorded interviews were conducted during 1984-85 with children in grades 4-6, enrolled in a Saturday morning mathematics program for the academically talented at Montclair State College. The interviewers were graduate students in mathematics education at Rutgers University; most were also elementary and secondary school teachers. The purpose of the script was to prescribe (to the extent possible, verbatim) the clinician's role in such an individual interview. The clinician guides the child through a complex plan, applying TSP to solve the problem:

> "What is the remainder when two to the 50th power is divided by three?"

The major steps in the plan are the following: (a) understanding the original problem, and recognizing that it does not yield to direct computation; (b) deciding to construct and solve one or more simpler, related problems; (c) generating and solving a sequence of simpler related problems, in which the exponent in the original problem is changed successively from 50 to 2, 3, 4, etc. (the special case in which the exponent is 1 is omitted as a difficult instance of the concept of "power"); (d) detecting the pattern which occurs; (e) making a table of the remainders, if necessary, and conjecturing a value for the remainder in the original problem based on the pattern; and (f) when feasible, finding a "reason" for the pattern. The script based on this plan is presently 19 pages long, and typically requires about 30 to 45 minutes per interview to administer. It is still undergoing refinement and revision. An outline of its major sections is presented in Table 1.

Table 1. Outline of the Script for "Think of a Simpler Problem" (TSP)

I. Explanation of prerequisites for understanding the problem

   a. the meaning of 3 to the 4th power, and similar problems;

   b. the remainder when 17 is divided by 4, and similar problems;

   c. the remainder when 3 to the 2nd power is divided by 5, and similar problems.

II. Presentation of the main problem

   a. posing the problem of finding the remainder when 2 to the 50th power is divided by 3;

   b. determining if the child is willing to tackle the problem;

   c. observing the heuristic process or processes the child spontaneously uses;

   d. observing whether the child spontaneously decides to think of a simpler problem, and if so, observing in detail how far the child takes the process (see below);

   e. observing other noteworthy occurrences, including expressions of affect;

   f. noting whether the child states a coherent reason for what he or she did.

III. Guided use of the heuristic process TSP (if the child has made no prior use of this process, we continue here without "correcting" any prior conceptions or misconceptions)

   a. observing whether the child seeks to think of a simpler related problem when prompted to do so; if so, observing whether the child successfully generates one;

   b. if not, observing whether the child does so when suggestions of how to obtain a simpler problem are made;

   c. observing other noteworthy processes.

IV. Presentation of simpler problems (if simpler problems have already been obtained, we enter this section at the appropriate place)

   a. suggesting the child try, "What is the remainder when 2 to the 2nd power is divided by 3?" and guiding the child to its solution;

- b. suggesting the child try the main problem again; observing if the child now spontaneously generates additional simpler problems; cycling back through III.b, increasing the exponent in IV.a by 1 each time;
- c. observing the point where the child spontaneously generates simpler problems, and/or detects a pattern in the remainders;
- d. observing other noteworthy processes and expressions of affect.

V. Guided detection of the pattern in the remainders (if not previously found)

- a. observing whether the child detects a pattern when prompted to look for one;
- b. if so, noting whether the child spontaneously infers or conjectures a solution to the main problem;
- c. if not, guiding the child to detection of the pattern and observing whether a solution to the main problem is inferred;
- d. observing other noteworthy processes.

VI. Guided conjectured solution to the main problem using the pattern in the remainders

- a. observing whether the solution is conjectured by the child when prompted;
- b. if not, guiding the child through construction of a table and observing whether solution is conjectured;
- c. if necessary, guiding the child to conjecture a correct solution to the original problem;
- d. observing whether the child recognizes spontaneously that the solution is only a conjecture, and/or looks for a "reason" why the pattern occurs;
- e. observing transfer to related problems.

VII. Looking back

- a. noting the child's feelings about the problem;
- b. noting the child's recognition of previously-encountered similar problems;
- c. asking for a coherent retrospective account, asking for alternate methods or shortcuts;
- d. correcting conceptual misunderstandings which may have occurred during the problem-solving interview.

The following is a guiding principle applied throughout the script which, in my opinion, is crucial to its validity in permitting us to make inferences about children's cognitions: Whenever a question is asked or a suggestion made, the child is permitted to work freely until he or she arrives at a conclusion or an impasse. At the outset (Sec. I), the concept of "power" is reviewed or introduced, and the child is asked to explain it. If necessary, the clinician illustrates its meaning, assisting the child until examples are solved correctly. When the main problem is introduced (Sec. II), if it seems that it is being totally misinterpreted, the clinician asks, "Could you explain what two to the 50th power means?" and corrects the child's understanding of the meaning of the problem if necessary. However, the clinician does not at this time correct conceptual misunderstandings or misapplications of arithmetical rules, such as the assumption that two to the 50th power is the same as 50 to the second power. The child does not receive feedback correcting conceptual misunderstandings until the very end of the interview--even when the child's conclusions at a particular point reflects such a misconception, the clinician simply proceeds with the next suggestion in the script. Thus the child has the opportunity either to generate a more adequate conceptualization (through the application of the heuristic process), or to bypass the original misconception and solve the problem in a different way. The child is led to self-correct computational errors only after observation to see that this does not take place spontaneously, and only when necessary to implement a required step in the overall plan.

PROTOCOL ANALYSIS AND SCORING

The script provides for various response alternatives--for example, if the child spontaneously conjectures a correct answer to the original problem, based on the pattern in the sequence of remainders, he or she is not led to construct a table. Thus not every subprocess associated with TSP is encountered in every interview. But for each subprocess encountered, transcripts of the tape-recorded interviews, together with the child's worksheets, permit an assessment of competence: (1) Does the child use or attempt to use the process spontaneously? (2) Does the child use or attempt to use the process when prompted to do so? (3) Is the child's (spontaneous or prompted) application of the subprocess successful? Thus it is possible to measure, for instance, a child's ability to generate spontaneously a sequence of simpler related problems (a) when presented with the main problem (Sec. II), (b) when prompted to think of a simpler related problem (Sec. III), (c) when a specific simpler related problem is presented (Sec. IV), and so forth. More details of the script and its proposed scoring are provided in an accompanying paper (Goldin and Landis, 1985).

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A PROBLEM SOLVING INTERVIEW WITH STAN (AGE 11)

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The interview script developed by Goldin, based on the heuristic process "think of a simpler problem," is described in an accompanying paper. This script was used in a pilot study with children in grades 4-6, enrolled in a Saturday morning mathematics program for the academically talented at Montclair State College. We analyze some of the problem-solving processes of "Stan," an eleven-year-old boy, as they were recorded in the interview, illustrating how a child's competencies in various subprocesses of the heuristic process can be observed and scored.

BACKGROUND

In an accompanying paper, Goldin (1985) reports on the creation of a guided discovery script for conducting individual clinical interviews with academically talented children, in order to study their competencies in subprocesses of a complex heuristic process, "think of a simpler problem" (TSP). In the Fall of 1984, versions of this script were administered in a pilot study to children in grades 4-6, enrolled in a Saturday morning mathematics program for the academically talented at Montclair State College. One boy, "Stan" (not his real name), who was 11 years old at the time of the interview, displayed some interesting and remarkable insights as he solved the problem that was posed. This paper describes some of the problem-solving processes that he employed. It is intended to illustrate how a child's competencies in various subprocesses can be observed and scored in the context of an extended and complex problem-solving activity.

PROCEDURE

Graduate students in mathematics education at Rutgers University, many of whom were also K-12 teachers, were trained to act as clinicians in administering the script. Early drafts of the script were read and criticized, and revisions were made based on the students' suggestions. The students then rehearsed the script in pairs, interviewing each other and taking turns playing the child's role. Further revisions of the script took place as rough spots were uncovered during these sessions. Finally each student, now familiar with the script, administered it to a child; the tape of this interview was replayed, and the clinician's adherence to the script discussed. After such practice and discussion, the pilot study at Montclair State College began in November 1984. The process of revision of the script continued for several weeks during the ensuing interviews.

In each interview, the child was provided with pencils and paper for use during the problem solving. One graduate student functioned as the clinician, sitting beside the child, posing the problem in accordance with the script, and making notes on a copy of the script. A second graduate student functioned as an unobtrusive observer, tape recording the interview and following along, making notes on a copy of the script. The observer was permitted to ask questions only at the very end of the interview. After the interview, the tapes were duplicated and transcribed. A file for each child thus consisted of the tape of the interview, a verbatim transcript of the tape, the child's actual worksheets, and scripts with notes made during the interview by the clinician and by the observer. J. Landis was the clinician for the interview with Stan, who was then in the 6th grade.

HIGHLIGHTS OF STAN'S PROTOCOL (ABRIDGED)

Stan had previously learned about exponents in school, and had no difficulty with the prerequisites (Sec. I of the script). When asked to find the remainder when 2 to the 50th power was divided by 3 (Sec. II), he did not write out an expression using 2 as a factor 50 times, nor did he begin to calculate products.

(Stan:) Okay. First I have to figure out 2 to the 50th power, and I don't remember a short way of doing it. Okay. Should I do the whole problem? (Clinician:) Try and think out loud, tell me what you're thinking as you're doing it, okay? (S:) I'm trying to find a shorter way to doing 2 used as a factor 50 times, and I'm thinking that 50 might be able to be done to the 2nd power; just reverse it, and that would be 50 times 50, that would be 2,500 divided by 3, that would give me ... the remainder would be one. (C:) Okay, can you tell me why you did what you did? (S:) Because I think that 2 to the 50th power, which is 2 used as a factor 50 times, you would just reverse the numbers, that would be 50 used as a factor 2 times.

Note that Stan did not spontaneously monitor for the correctness of his generalization, for example by trying a special case. He had arrived at an answer to the problem, and was satisfied with his conclusion. Therefore, without correcting his "overgeneralization" of the commutative property, the clinician next prompted him (Sec. III):

(C:) ... Can you think of a simpler problem that might help you to solve this one? (S:) Umm ... 2 to the 25th or 2 to the 12th, maybe. (C:) Okay. Which one would you like to use? (S:) Umm ... I'll do to the 6th, which is half of that. ... And I got the answer is 64, and 64 ... I got the remainder would be one, again. (C:) ... will that help you solve the original problem? (S:) Umm. I don't really think so, because 6 does not go into 50. ... I could try 5, because that goes into 50. ... That would be 32, and 32 divided by 3 is ... the remainder would be 2. Actually, ... that would be 10 remainder 2. (C:) ... Now would that help you if you were trying to find the remainder of 2 to the 50th divided by 3? (S:) Yes. I think so, because I would take my answer and try and work it into 50. ... Like, okay. Two to the 5th is, um, 32, so if I wanted to get to the 50th, I could multiply it by 10, which would give me, 5 times 10 is 50. That would be 2 to the 50th, and that would be 32 times ten, 320, divided by 3, is one ... (computes) ... the remainder would be 2.

We see that Stan successfully generated simpler problems when prompted to do so, and formulated a plan for using them. This time he "overgeneralized" the property of associativity and, as with the commutative property, he did not monitor for the correctness of his generalization. Prompted now to think of another simpler problem that could help him (cycling back through Sec. III), Stan suggested 2 to the 25th ("which would be even closer to 50"), but changed his subgoal spontaneously:

(S:) Okay. Two to the 6th equals 64 ... I want to get to the 25th ... 2 to the 10th is 1,024 and I could use 10 to get into 50. ... Now I have to ... I could ... hmm .. I could maybe divide that by 3 and multiply the answer by 40, I mean, by 5, that would give me 50 after I divided, and then I wouldn't have to divide at the end and that would be ... the remainder would be one.

Here Stan conjectured in effect that $2^{50} \div 3 = (2^{10} \times 5) \div 3 = (2^{10} \div 3) \times 5$. Though he seemed slightly dubious about his procedure, he still did not test for its correctness, or note the contradictions among his answers to this point. His mention of multiplying by 40 may merely reflect momentary confusion of division with subtraction, or it may reflect a fleeting intuition of the law of exponents. Since Stan generated many simpler problems when prompted, but did not spontaneously look for a pattern in the remainders he found, the clinician continued with Sec. IV of the script:

(C:) Okay, let's suppose we tried 2 to the 2nd power divided by 3. What would that remainder be? (S:) Umm, one. (C:) Do you have any new ideas now for solving the problem of the remainder when 2 to the 50th power is divided by 3? (S:) Umm, yes, I could try and find a pattern. (C:) What do you mean? (S:) Like I would try 2 to the 3rd, and 2 to the 4th, and 2 to the 5th, and if the remainder was the same in each one, I think it would be safe to assume that it would be all the way up to 50 ... but, it doesn't work. Well, actually the remainder would be 2 in that ... the remainder would be one ... 2 ... okay, I found a pattern. (C:) You did. What did you find? (S:) In 2 to the 2nd the remainder is one, 2 to the 3rd it's 2, 2 to the 4th it's one, 2 to the 5th it's 2, 2 to the 6th it's one and it keeps on going. And all the exponents are even when the remainder is one, therefore, the remainder would be one in 50, because 50 is an even number.

Stan spontaneously articulated his plan to look for a pattern in the answers to a sequence of problems. Though he did not find the pattern he anticipated, he found one that enabled him, without further prompting, to infer a solution to the main problem. Thus Secs. V and most of Sec. VI, guiding the child through these steps, do not appear in scoring Stan's competencies. Asked if this problem reminded him of any others he had solved before (Sec. VII.b), Stan described in detail a problem from his class at Montclair State College, where each of 1000 students in turn opens, closes, or leaves unchanged 1000 lockers--student 1 opens all the lockers, student 2 closes the even-numbered ones, student 3 reverses every 3rd locker, etc.

(S:) ... And the problem was which lockers would remain open. And you had to figure it out. I did all the way up to 20, and I noticed that the numbers that were opened in the 20 were 1, 4, 9 and 16. When I looked at those numbers, I realized that those numbers were perfect squares, and 1 would be one times one, 4 would be 2 time 2, 9 would be 3 times 3, and I also noticed another pattern. Uh, the difference between 1 and 4 is 3, the difference between 4 and 9 is 5, the difference between 9 and 16 is 7, and it keeps going up by 2, all the way to 961, and that was the highest number, and it turned out that there were 31 lockers that would be open. And I listed all of them.

The resemblance Stan recognized between the two problems was based not on any surface similarity in syntax or context, or structural similarity in required arithmetic operation; rather it was based on the similar applicability of a process of constructing a sequence and detecting a pattern in it. Stan then gave a coherent retrospective account (Sec. VII.c) of his discovery of a pattern in the remainders, and at that point sought (successfully) to articulate a reason for the pattern.

SCORING OF COMPETENCIES

For each subprocess of TSP, at each stage of the problem-solving interview, we seek to determine whether the child uses the subprocesses: (1) spontaneously, (2) when prompted to do so,

(3) successfully. Table 1 reports the scoring of some of the subprocesses evidenced in the interview with Stan.

Table 1. Partial Scoring of Stan's Competencies(a)

(a) Roman numerals refer to the script outline presented in the accompanying paper (Goldin, 1985).

	successfully	spontaneously	when prompted	not at all
II. Presentation of the main problem				
Decides to seek a simpler method	no	x		
Monitors for correctness of generalization				x
Conjectures solution to original problem	no	x		
Recognizes solution as merely conjecture				x
Decides to think of simpler problem				x
III. Guided use of the heuristic process TSP				
Decides to think of simpler problem	x		x	
Solves simpler problem	x	x		
Generates sequence of related problems	x	x		
Looks for a pattern in the sequence				x
Conjectures solution to original problem	no	x		
IV. Presentation of simpler problems				
Generates sequence of related problems	x	x		
Looks for a pattern in the sequence of simpler related problems	x	x		
Conjectures solution to original problem based on pattern in sequence	x	x		
Recognizes conjectured solution as merely conjecture, seeks reason behind pattern				x
VI. e. Observing transfer to related problems				
Applies the pattern to $2^{44} \div 3$	x	x		
Applies the pattern to $2^{75} \div 3$	x	x		
Describes application of TSP to $3^{50} \div 4$	x	x		
VII. Looking back				
Recognizes previously solved problem related by heuristic process	x	x		
Provides coherent retrospective account	x	x		

Reference: Goldin, G.A. (1985) Studying Children's Use of Heuristic Processes for Mathematical Problem Solving Through Structured Clinical Interviews. To be published in Procs. of the 7th Annual Mtg. of PME-NA.

METACOGNITION RESEARCH: TECHNIQUES OF PROTOCOL ANALYSIS

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ABSTRACT

Some mathematics educators have taken the position that metacognition (knowledge and beliefs about cognitive activity as well as awareness and control of that knowledge) is critical to mathematical performance and that research in mathematical problem solving would benefit from exploration of this phenomenon. The goal of the current work was to explore the use of Schoenfeld's (1983) framework to recognize, record and analyze observable metacognitive activity during mathematical problem-solving sessions. The protocol of three preservice middle school teachers solving an applied mathematical problem was studied. Several factors were found to influence our analytical process. These included number of members in the group, expertness of the problem solver, and type of problem.

BACKGROUND

In the last few years much attention has been given to metacognition in research on cognitive development, memory, and reading. Mathematical problem-solving researchers, however, have just begun to question the impact metacognitive activity may have on problem-solving success and to offer a clear definition (Lester & Garafalo, 1985) of what is meant by the term. Some mathematics educators (e.g., Lesh, 1982; Lester & Garafalo, 1985; Schoenfeld, 1983; Schultz & Hart, 1985; Silver, 1984) have taken the position that knowledge and beliefs about cognitive activity as well as awareness and control of that knowledge is critical to mathematical performance and that research in mathematical problem solving would benefit from exploration of this phenomenon.

GOALS OF THE STUDY

The short-term goal of the current work was to develop methods and approaches to recognize, record, and analyze observable metacognitive activity during mathematical problem solving. Using a framework developed by Schoenfeld (1983), it was our intention to describe the process of recognizing, recording, and analyzing this phenomenon as it actually developed, with an eye for sharing our data-gathering and interpretive procedures with others. Our long-term goal is to ascertain a correlation between problem-solving success and metacognitive activity.

THE STUDY

Data Source

Data for this study were videotapes and transcripts from pilot studies (Lesh & Schultz, 1983), where the problems solved were applied problems taken from Lesh's (1982) applied Mathematical Problem Solving Project. One protocol of a group of three preservice middle school teachers was selected for analysis.

Procedure

The process for the transcript began by parsing the protocol into episodes of six different types: reading, analysis, exploration, planning, implementation, and verification. A brief synopsis of Schoenfeld's (1983) description of each episode type follows.

Reading. A reading episode of a problem-solving protocol starts when a subject reads the problem aloud and continues on through preliminary ingestion of the problem conditions and any silence immediately following--implying nonverbal rereading. It may include assessment of the current state of the problem-solver's knowledge related to the problem-solving task.

Analysis. An analysis episode occurs when there is no obvious solution path after a reading episode. In this kind of episode the subject makes an attempt to understand a problem fully, to select an appropriate perspective and reformulate the problem in those terms. Analysis sticks rather closely to the conditions and goals of the problem.

Exploration. An exploration episode is less structured than an analysis episode and is further removed from the original problem. It frequently includes a variety of problem-solving heuristics. Schoenfeld describes it as "a broad tour through the problem space" (p. 358).

Planning-Implementation. Planning and implementation are self-explanatory. Planning, however, is not always evident in a problem solution. If a problem solver moves directly into "doing" the problem, the episode is identified simply as implementation. Where planning is evident as a separate activity it is classified as a planning episode. When both planning and implementation seem to be occurring simultaneously an episode is identified as planning-implementation.

Verification. Verification involves assessing, reviewing, and testing a solution and evaluating the solution process.

After parsing, a problem-solving protocol is mapped in a flowchart-type of diagram from episode type to episode type with notations at transitions between episodes. The introduction of new information and local assessment of progress within an episode as well as transitions between episodes are critical points. It is at these three points that metacognitive activity may be observed.

The protocol was analyzed by two doctoral students and ourselves. After lengthy discussion on interpretations of episode types, we developed fairly high consistency in parsing.

Discussion

As with any model in its formative stages, Schoenfeld's model presented some problems for us--given our long-term research goal of ascertaining a correlation between problem-solving success and metacognitive activity. The most obvious problem was number of persons in the problem-solving session. While two- and three-person groups may be quite useful in some research settings (e.g., Nodding [1983] and Hart [1984] found that average and below-average subjects learned problem-solving skills while working in a group), for this type of analysis it produced certain difficulties. The most obvious one was when members of the group were actually operating in different episode types. Consider an excerpt from the videotape of Caran, Magda, and Chuck solving Lesh's (1982) Carpentry Problem. The problem is presented in parts. The first part asks the solver to determine how much baseboard is needed for a 21-foot by 28-foot room if baseboard comes in 16- and 10-foot sections. The group answered this question by multiplying length times width to obtain 588, dividing it by the length of the longer board which was 16, and concluding that they needed 37 16-foot boards. The second part of the problem asks the problem solver to calculate the baseboard needed to have the fewest seams. In attempting to answer the problem with this new stipulation, Caran discovers that they have made an error in the earlier solution for part one. She and Magda begin implementation of a new plan. Chuck, however, has already worked out a solution (which finds the least waste, not the fewest seams) and he continually tries to offer it to the group. It is unclear from the video or the transcripts at what point Chuck planned and implemented his solution and what metacognitive activity may have monitored his work. This excerpt is presented below.

Caran: So what's 28 divided by--oh, you know what, you guys?

Magda: Huh, what? What?

Caran: We figured it out, this is--this 588 is area, not perimeter. That's what nervousness will do for you.

Magda: Okay, so--

Caran: Cause when I started figuring out this I was going how come we only need 1.75 boards and we've got 36 we're buying?

Magda: Well, let's start over then.
Caran: That's great.
Magda: We'll just have to slow down.
Caran: Right [calculating]--42.
Magda: Okay.
Caran: Oh, God.
Chuck: I'll tell you what we need. I wrote it down.
Magda: Let's see--no, we need--
Chuck: 98 feet of board is what we need.

In analysis of the protocol the group was said to be in an implementation episode at the end, but Chuck had already developed a plan, implemented his plan, and had a solution to offer to the group. Using group protocols we were not able to isolate individual metacognitive activity.

Our intention for future work is to modify the format so we can more closely monitor individual metacognition. We propose to do this by videotaping teachers solving totally unfamiliar problems before a group of students. By being in the teaching setting subjects will be forced to think aloud in order to model problem-solving behavior. In this way we will obtain the thought processes of an individual problem solver. These sessions will be videotaped and analyzed for metacognitive activity, using the Schoenfeld model.

Another problem is one suggested by Schoenfeld (1983) in his description of his technique.

> There are both objective and subjective components to the framework for analyzing protocols. The objective part consists of identifying in the protocols the loci of potential managerial decisions. The subjective part consists of characterizing the nature of the decision points and describing the impact of those decisions (or their absence) on the overall problem-solving process. (p. 354)

We found the objective analysis quite challenging. An issue such as interpretation of episode typing is initially critical in developing consistent parsing. Other factors that influence both the objective and subjective components of the analysis must be considered. Some of these are novice/expert problem solvers, number of members of the group, and problem type. Careful examination and documentation of all these factors must be considered in light of the research questions being raised.

The subjective analysis is still not resolved in our minds. We were too quick to try to quantify the presence of metacognitive activity with a score at critical points. Expert problem solvers may show little metacognitive activity and come to a quick and elegant solution of the problem, whereas novice problem solvers might show extensive metacognitive activity and never arrive at a solution. The amount of metacognitive activity therefore is not necessarily a predictor of problem-solving success. A score is a function of the expertness of the problem solver as well as the problem type, as well as other factors. Our inclination at this time is to observe the ratio of productive critical points to total critical points for a single problem solver and to work toward identifying types or levels of metacognitive activity into which problem solvers may be placed. One control for the expert/novice factor is to be certain the problem we give is really a problem for the individual--certainly not an easy goal.

SUMMARY

We have no definitive conclusions about the analysis procedures at this point. It is expected that our interpretations and characterizations will be cyclic in their development. An initial conceptualization must be filtered, organized, and interpreted through several phases of refinement before it becomes a more usable model. We feel confident in our progress and encourage reaction and input to our work.

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THE EFFECT OF RATE TYPE, PROBLEM SETTING AND RATIONAL NUMBER ACHIEVEMENT ON SEVENTH-GRADE STUDENTS' PERFORMANCE ON QUALITATIVE AND NUMERICAL PROPORTIONAL REASONING PROBLEMS

The Pilot -

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ABSTRACT

This study investigated how different rate types (buying, speed, consumption) and problem settings (familiar and unfamiliar) affected 254 seventh graders' performance on qualitative and numerical proportional reasoning problems. Six forms of a proportional reasoning test were designed reflecting the three rate types and two settings. Each form consisted of 17 questions in a single context and included missing value and numerical comparison word problems, 1 non proportional reasoning word problem and 10 qualitative reasoning problems. Each student was given one of the 6 forms and a 20 item rational number test. Rate type and rational number ability affected student achievement on both numerical and qualitative subscales of the proportional reasoning tests. As expected the less familiar rate (consumption) was more difficult for both subscales. Correlations between different parts of both tests were moderate to low suggesting that students do not perceive that rational number concepts and proportional reasoning skills are in fact related to one another.

We are indebted to Nadine Bezuk, Kathleen Cramer and Andrew Ahlgren who assisted in this research. The research was supported in part by the National Science Foundation under Grant No. DPE-847077. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The purpose of this study was to investigate how different rate types and problem settings affect student performance on qualitative and numerical proportional reasoning problems. In addition, relationships between qualitative directional reasoning about rates and numerical proportional reasoning, and relationship between rational number skills and numerical proportional reasoning were explored. The following questions were posed:

1. Does the type of rate differentially affect student performance on qualitative and numerical proportional reasoning problems?
2. For a given rate type, do differences in problem settings affect student performance on qualitative and numerical proportional reasoning problems?
3. To what extent is qualitative directional reasoning about rates related to numerical proportional reasoning?
4. To what extent are rational number skills related to student performance on proportional reasoning problems?

Numerous studies have shown that early adolescents and many adults have a great deal of difficulty solving proportional reasoning problems (Hart, 1978, 1981; Karplus, 1981; Karplus et al., 1979; Rupley, 1981; Suarez, 1977; Vergnaud, 1980, 1983).

Why is proportional reasoning so difficult? What factors affect problem solving success? Several studies have shown that factors such as problem format, the numerical characteristics of the problems, the problem context, and even the immediately preceding problem affect student performance on proportional reasoning problems (Jesunathadas and Saunders, 1985; Karplus et al., 1984; Lybeck, 1978; Rupley, 1981; Vergnaud, 1980). In this study we investigate the effect of two aspects of problem context on the level of student performance on qualitative and numerical proportional reasoning problems. The intent eventually is to explore the practicality of using a graded series of exercises to lead students from proportions that are fairly easy to understand to those more difficult proportions essential to the sciences (e.g., density, acceleration, concentration, definite proportions, genetics, etc.) and to more advanced mathematical applications, i.e., algebra.

Several studies have shown that student performance on proportional reasoning problems is affected greatly by the problem context (Karplus et al., 1983; Lybeck, 1978; Vergnaud, 1980). Jesunathadas and Saunders (1985) found that familiarity with the content of proportional reasoning tasks affected ninth-grade students' performance on these tasks. Students had significantly greater success solving problems with familiar content than solving problems that were the same numerically but with unfamiliar science content. Familiar content was defined as those words, processes, and concepts which most students encounter quite frequently in their daily lives. Unfamiliar science content was defined as those words, processes, and concepts which are found in high school science textbooks.

There may be two aspects of "context" that can be usefully distinguished in proportional reasoning problems. The first has three sub-considerations: (a) the objects in the problem, (b) the variables used to describe the two properties of the objects of interest in the problem (e.g., length, area, weight, time, etc.), and (c) the units of measurement used to specify these variables (e.g., for length--inches, feet, centimeters, kilometers, miles, etc.). We will call this set of context aspects the "problem setting." For example, in speed problems students may be more familiar with people running foot races and measures of distance run in laps and running time in minutes than they are with driving cars and measures of distance traveled in miles and driving time in hours.

A second aspect of context is the type of ratio or rate involved in the problem. A survey indicated eight types of rates that can be found in standard textbook proportional reasoning problems. They are: Distribution (cookies per person), Packing (books per foot on shelf), Exchange (dollars per hour), Mixture (orange juice concentrate and water), Speed (nails hammered per minute), Consume or Produce (miles traveled per gallon), Scale (inches per mile) and Conversion (points per kilogram). Most can be interpreted with direct or continuous variables. Each type of rate can be used in familiar or unfamiliar problem settings. Even with familiar problem settings, however, students may be more or less familiar with the rate types themselves. For example, junior high school students typically have more experience buying or mixing than they have scaling or converting units of measurement.

Familiarity with what is called the problem context may consist of familiarity with both the rate type and the problem setting. Knowledge of the hierarchy of difficulty for uninstructed students on proportional reasoning problems with different rate types and problem settings may contribute to a better understanding of how proportional reasoning skills develop in adolescents and to the design of better proportional reasoning instruction for students.

Another factor which could affect student performance on proportional reasoning problems is qualitative reasoning skills, which seems to be a significant variable in mathematics and physics problem solving performance (Chi, Feltovich and Glaser, 1981; Larkin and Reif, 1979; Larkin et al., 1980). Some proportional reasoning studies indicate that many early adolescents use faulty qualitative reasoning or use additive comparisons where multiplicative comparisons are required (Karplus and Peterson, 1970; Karplus et al., 1983; Noelting, 1980 a & b). The frequency of these incorrect strategies seems to depend on the problem context (Jesunathadas and Saunders, 1985; Karplus et al., 1983). However, no systematic research has been conducted to explore students' ability to reason qualitatively about rates, to determine the effect of different contexts on their qualitative reasoning about rates, or to determine how qualitative reasoning about rates contributes to proportional reasoning skills.

In this study we introduce a new type of qualitative question about rates that may be important in understanding the development of proportional reasoning skills in adolescents. These questions ask in what direction a rate will change (decrease, stay the same, or increase in value) when the numerator and/or the denominator decreases, stays the same, or increases. Such qualitative directional reasoning about rates may be important prerequisite skill for successful performance on numerical proportional reasoning problems.

The presence of integral ratios or rates in proportional reasoning problems and small numerical values less than about 30 make problems considerably easier than those without integral ratios or larger numbers (Karplus et al., 1983; Noelting, 1980 a,b; Rupley, 1981). It would seem, then, that rational number skills could be an important prerequisite skill for successful performance on proportional reasoning problems.

In this study, we limited our investigation of numerical proportional reasoning to problems with easy, integral ratios or rates. Since we were interested in the effect of rate type and problem setting on problem solving performance, we did not want to add a numerical difficulty interaction effect. We did, however, examine the relationship between rational number skills and performance on numerically easy proportional reasoning problems.

Three types of rates were examined in this study: exchange rates (buying), speed, and consumption rates. The two problem settings selected for each rate type were (a) Buying - gum and records, (b) Speed - running laps and driving cars, (c) Consumption - gas mileage of trucks and oil burning in furnaces. These rate types were chosen because we expected them to have different difficulties and because they have been studied previously (Karplus et al., 1983; Vergnaud, 1983). We expected speed problems to be slightly more difficult than buying problems, and consumption problems to be the most difficult of the three rate types.

Numerical Proportional Reasoning Problems

> Two formats of numerical problems, missing-value and numerical comparison problems, were used in this study, as illustrated by the problems below:
> Missing-Value:
>
> Steve and Mark were running equally fast around a track. It took Steve 20 minutes to run 4 laps. How long did it take Mark to run 12 laps? Please show all your work carefully.
>
> Numerical-Comparison:
>
> Tom and Bob ran around a track after school.
> Tom ran 8 laps in 32 minutes.
> Bob ran 2 laps in 10 minutes.
> Who was the faster runner?
> ____ Tom ____ Bob ___ they ran equally fast ___ not enough information to tell

Missing-value and numerical-comparison problems have been used extensively in instruction and research. The inclusion of both types of problems in this study complements previous studies with the same rate types by Karplus et al. (1983) and Vergnaud (1980). The numerical values chosen for each type of problem are contained in Table 1 below.

Table 1
Data Used in Numerical Problems

	Person 1		Person 2	
	V_1	V_2	V_1	V_2
Missing-value 1	4	20	12	?
Missing-value 2	8	24	?	6
Missing-value 3	3	12	?	36
Comparison 1	8	32	2	10
Comparison 2	3	15	9	45
Comparison 3	6	24	2	6

* V_1 and V_2 are the two variables in the problem setting (for example, number of pieces of gum and price in cents).

Qualitative Proportional Reasoning Problems

Two formats of qualitative directional questions about rates were invented for this study, as illustrated by the questions below:

Qualitative Rate Change:

If Nick ran less laps in more time than he did yesterday, his running speed would be

a) faster
b) slower
c) exactly the same
d) there is not enough information to tell

Qualitative Comparison:

Bill ran the same number of laps as Greg. Bill ran for more time than Greg. Who was the faster runner?

a) Bill
b) Greg
c) they ran at exactly the same speed
d) there is not enough information to tell

Since both the numerator and the denominator of a rate can decrease, increase, or remain the same, there are nine qualitative rate change and nine qualitative comparison questions that could be asked. Two cases are ambiguous. Ambiguity occurs when the numerator and denominator both increase, or both decrease. The correct answer to these questions is that there is not enough information to tell what happens to the value of the rate (qualitative rate change) or which object has the larger value of the rate (qualitative comparison), because the numerator and denominator can decrease (or increase) proportionally or non-proportionally. These are the only qualitative questions that require a truly numerical understanding of proportionality for a correct answer.

METHODS

Subjects

Our subjects were 254 seventh graders in a middle-class urban school in St. Paul, Minnesota. They included all seventh-grade students in attendance on the day the tests were administered. About half of each group were girls and about half were boys. The teachers reported that the students had not received instruction on proportional reasoning problems in their seventh-grade mathematics classes.

The Instruments

Six forms of the proportional reasoning test were designed, each comprising 17 questions in a single context, three rate types--two settings within each. The first section of the proportional reasoning test consisted of three missing-value and three numerical-comparison problems. The numerical values in the six problems, shown in Table 1, allow students to solve the problems correctly using integer ratios or rates. The second section of the test contained qualitative questions similiar to those already described. One item did not involve proportional reasoning.

The second instrument used in the study was a 20-item rational number test. This test consisted of problems or order and equivalence, finding equivalent fractions, qualitative changes in the value of a fraction, operations with fractions, estimating rational number computations, a quantitative notion of a fraction, and the concept of a unit. The test was constructed so as to correspond numerically to the numbers used in the proportional reasoning test.

Procedure

The tests were administered according to a set of instructions which was read and explained to the students. The six different forms of the proportional reasoning test were randomly distributed to the students in each class. After each student completed the proportional reasoning test, he or she was given the rational number test.

ANALYSIS

Table 2 contains means and standard deviations for the three rate types, and two settings within each for the numerical (missing value plus numerical comparison) and for the qualitative problems.

TABLE 2
Means and Standard Deviations

Type of Rate	Numerical (possible Score = 6) Familiar Setting	Numerical (possible Score = 6) (Less Familiar Setting	Qualitative (possible Score = 8) Familiar Setting	Qualitative (possible Score = 8) Less Familiar Setting
Buying	4.23 (1.70) gum	4.37 (1.64) records	6.85 (1.48) gum	6.53 (2.21) records
Speeds	3.95 (1.78) laps	3.40 (1.80) cars	6.28 (1.75) laps	5.35 (2.26) cars
Consumption	3.53 (1.89) car/gas	2.97 (1.44) furnace/oil	6.00 (2.18) car/gas	4.00 (1.96) furnace/oil

*Means (Stantard Deviations) - i.e., Mean = 4.23; Standard Deviation = 1.70.

Students were divided into 3 roughly equivalent groups on the basis of their scores on the rational number test. Separate 3-way ANOVAS [rational number ability (3 levels), rate type (3 levels), and setting (2 levels)] were conducted for students' numerical and quantitative scores. Significant main effects for rational number ability and rate type were significant ($p < .001$) for both types of scores. Setting was significant only for the qualitative score. No significant two- or three-way interactions were observed. Figure 1 depicts the plot of the mean scores for each rate type and setting within each rate type.

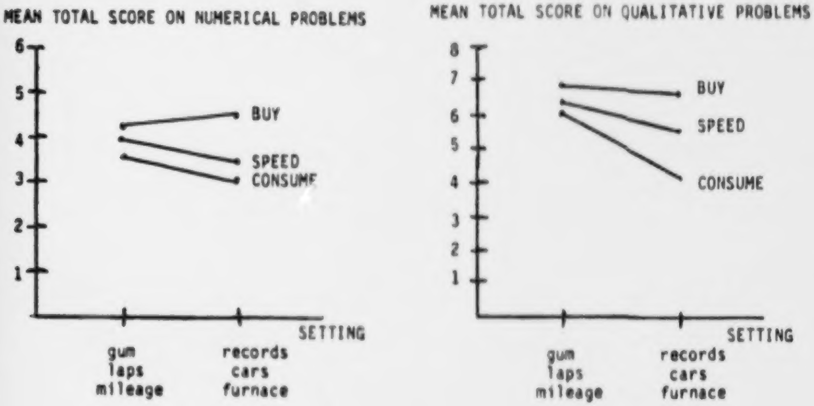

Figure 1

As expected, the less familiar rate (consumption) was more difficult for both the numerical and qualitative scales.

Correlations between the numerical (missing value plus numerical comparisons) and qualitative subscales scores on the proportional reasoning test and achievement on the rational number test were $r = .49$ and $r = .35$ respectively. Although these were significant at the .00001 level, the small percentages of variance accounted for, .24 and .12 respectively suggests that, in the latter case, students do not perceive that rational number concepts and the proportional reasoning skills measured by these tests are in fact highly related to one another.

There were very substantial achievement differences on various test items. The range was 5 to 92 percent correct. The two most difficult were those qualitative items requiring a determination of the qualitative effect on the overall rates (increase, decrease, stay the same, or impossible to tell) when both the numerator and the denominator increased or decreased. In these two cases the resulting direction of change is indeterminate. The correct interpretation is, of course, dependent on the rate of change of the numerator and the rate of change of the denominator in relation to one another. Requiring relativistic thinking, these items may in the future provide valuable insight into students' ability to process information in proportional reasoning situations. It should be noted that these two items were not included in the statistical analyses reported here because they did not load on the main factor in a factor analysis which was conducted. Achievement levels on the more/more, less/less items were by far the lowest of all items on both tests.

Postscript:

The actual study (of which this was the pilot) was completed by The Rational Number Project in the spring/summer of 1985, with over 900 7th and 8th grade students utilizing four different rate types (mixture, speed, scaling, and density), 2 settings for each rate type, and a test of rational number concepts which closely paralleled the proportional reasoning tests. Similiar data for 100 preservice elementary teachers at the University of Minnesota were also included as part of this effort. Results are currently being incorporated into a series of papers.

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CONFLICTING FRAMES OF REFERENCE IN THE LEARNING OF ALGEBRA

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In constructing meaning for algebraic expressions, the learner experiences obstacles due to his existing arithmetic frame of reference which interferes with his construction of an algebraic framework. In experimenting a teaching outline aimed at overcoming this initial cognitive problem, we uncovered evidence of different types of conflict between the arithmetic and algebraic frames of reference: one month after instruction in algebra, students reverted back to their arithmetic framework. However, when jolted back to algebra, their answers changed significantly.

In dealing with the cognitive difficulties involved in the learning of algebra, Davis (1975) has identified two major obstacles encountered by the learner, that of a lack of a numerical referent ("How can you multiply by x when you don't know what number it is!") and that of a "name-process" dilemma by which an expression such as $x + 3$ represents both the process of adding 3 to x, as well as naming the result. Another cognitive obstacle has been identified by Collis (1974), the beginning student's inability to accept the "lack of closure" of such expressions. However, it was Marilyn Matz (1979) who perceived that the student's arithmetic background might, to some extent, interfere with his learning of algebra. She pointed out that since in arithmetic, concatenation (the juxtaposition of symbols) denotes implicit addition, both for place-value ($23 = 20 + 3$) and in mixed fractions ($4\frac{1}{2} = 4 + \frac{1}{2}$), this additive connotation might bring the novice to conclude when given that $x = 6$, that $4x$ must be 46 or , when given $x = -3$ and $y = -5$, that xy must be -8 .

Research funded by the Quebec Ministry of Education (FCAR grants EQ-1741,2923)

In a previous paper (Chalouh & Herscovics, 1983) we communicated results which supported Matz' observation. We gathered our data by interviewing 6 students who had not been exposed to any formal instruction in algebra. Asked to respond to the question "Can you tell me what 3a means to you?" , five of the six students interpreted 3a in terms of a subdivision label (e.g. third problem, first part), three students provided a place-value interpretation (e.g. 3a is 30 or 3a is 31 because a is the first letter of the alphabet), two students showed evidence of a first letter abbreviation (e.g. 3 ants). Of our six subjects, three of them supplied us with more than one interpretation. Asked what they would get by replacing the letter a with the number 2 , one student responded "I would put 3 times or plus 2...", four students responded with "32", while the sixth student wrote "3 - 2 , 3 x 2 , 3+2, 32" .

These results show the very natural tendency by the students of interpreting an algebraic expression new to them in terms of the only numerical frame of reference they possess at this point, that of arithmetic. But this does not constitute cognitive conflict. Different frames of reference can come into conflict in various circumstances. They can be considered as conflicting when the existence of one interferes with the learner's construction of a new one. This is very well illustrated by the student who does not feel he can multiply a letter and a number. To him, the letter does not as yet represent a number and his arithmetic framework forces him to instantiate, that is, to substitute a numerical value for the literal symbol. But conflict between different frames of reference is not restricted to epistemological obstacles encountered in initial learning situations. Indeed, after the consecutive acquisition of two different frames of reference, the first one may prove to be predominant and interfere with the use of the second one. Yet another conflict occurs when a lack of delineation between two frames of reference existing in the learner's mind prevents him from identifying which one is relevant in a given situation. This paper will report how in a teaching experiment dealing with the first kind of cognitive problem, that of constructing meaning for algebraic expressions, we found evidence of the latter two types of conflict between the students' algebraic frame of reference and their arithmetic one.

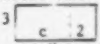
THE TEACHING EXPERIMENT

In order to introduce algebraic expressions in a way which might be meaningful to beginning algebra students we developed a new approach, one which sought to identify these expressions as "answers to problems". Since we wanted the students to focus on the algebraic expression, we selected problems of a highly pictorial nature, easy to visualize, which would not by themselves create cognitive obstacles. The types of problems we chose involved the quantification of a rectangular array of dots, the length of a line segment divided into parts, and the area of a rectangle.

The purpose of our teaching experiment was to uncover the possible new cognitive obstacles inherent in our teaching outline, and to assess whether or not differences in ability and grade level were relevant variables. Since we were interested in following the students' thinking, we opted for a case study approach. With each one of the six students mentioned earlier, we conducted a teaching experiment consisting of five semi-standardized interviews - a pre-test, three lessons, a post-test. Teachers selected three of them from grade 6 and from grade 7 of low, average and high mathematical ability as determined by their school performance. The reason for a semi-standardised format was to allow for inter-subject comparison.

Our first lesson started out very cautiously with our three types of problems, but each one involving a quantity hidden by a cardboard cover. In a prior paper (Herscovics & Chalouh, 1984) we reported how initially the students were introduced to the use of the placeholder box to represent the hidden quantity in completing statements such as "number of dots = 7 x $\square$, length = 4 x $\square$, area = 6 x $\square$". The removal of the cover exposed the hidden quantity which was then inserted in the placeholder box. Students had no difficulty in following this up by the use of letters instead of the placeholder box within the context of hidden quantities. As reported last year in a companion paper (Chalouh & Herscovics, 1984), our success with lesson 1 led us to believe that the transition to their use for the representation of unknown quantities (as in writing the area of $3\,\underset{a}{\boxed{}}$) would proceed without difficulty.

In fact, all six subjects indicated that they were experiencing some cognitive discontinuity in this transition. This led us to re-assess our impression that they were developing an acceptance for the lack of closure of algebraic expressions. Although this cognitive discontinuity was easily overcome, nevertheless it brought out the complexity and relevance of the name-process dilemma.

While the first part of lesson 2 dealt with the use of letters as specific unknowns, the second part of this lesson was devoted to concatenation. Students were introduced to the notation 3a as representing $3 \times a$ or $a \times 3$. We pointed out that this notation was unambiguous in algebra but could not be used in arithmetic since $3 \times 5 \neq 35$. It is in our third lesson that we started finding evidence of the algebraic frame of reference conflicting with the arithmetic one. When we asked students to write the area of the rectangle  for the small rectangle on the right-hand side, two of our subjects said "three times two" but wrote "32", one of them explaining that she meant "three two in algebra". However, it was the post-test that revealed the extent of this conflict.

THE POST-TEST

The post-test was administered one month after the third lesson was completed with all six subjects. It consisted initially of two parts: the first part repeated the questions raised in the pre-test in order to determine the change in our students' interpretation of algebraic symbolism; the second part contained questions intended to ascertain how much the subjects had learned from the three lessons. The first subject to be interviewed, Wendy, our average grade 6 student, surprised us by answering the questions in Part I almost identically as in the Pre-test, where she had used an arithmetic frame of reference. At the beginning of the second part of the post-test, when she was asked the meaning of the expression 5b , she questioned whether we wanted her to respond "in algebra". This chance remark by her led us to modify the post-test to include a third component assigned to review some of the questions in Part I, but specifically asking the students to answer in "algebra".

Here are some of the questions asked in both the pre-test and the post-test:

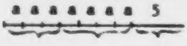
- When I show you this, $\boxed{3a}$, can you tell me what it means to you ?
- If you replace the letter a by the number 2, can you tell me what you get ?
- When I show you $\boxed{4 - b}$, can you tell me what it means to you ?
- Look at this workcard: $\boxed{\text{Simplify } 2a + 5a}$ Can you tell me what it means? Can you do it ?
- Can you simplify: $\boxed{3a + 4a + 5}$, $\boxed{2c + 3d + 4c}$ Why do you think different letters are used ?
- Can you add 4 onto $n + 5$? Can you add 4 onto $3n$?

Comparing the initial Post-test results with those of the pre-test, we found that our subjects' responses were almost identical, that is, our subjects were still essentially responding within an arithmetic frame of reference. We had expected that after the work done in the three lessons, the responses gathered in the post-test would have been significantly different and reflect an algebraic context.

The questions dealing with the concatenated expression 3a showed that four of the students answered as in the pre-test: Wendy our average grade 6 subject, thought of place-value and alphabetical rank (a=1, b=2) and wrote "3a = 31" ; Frankie, our weak grade 6 student tought algebraically, "3a is 3 times a" ; Gail, our strong 7th grader, said "three apples, three ants" and referred to place value ; Filippo, our average 7th grader, referred to place-value and to subdivision label. Two students demonstrated some evolution. Yvette, our weak grade 7 student, indicated that she had expanded her initial subdivision label and place-value interpretation of 3a by drawing a rectangle whose area was 3a. Antoinette, our strong 6th grader, was the only student showing a complete change from her initial arithmetic interpretations (subdivision label and place-value) to a purely algebraic one ("3a is 3 times a"). All students, except Frankie and Antoinette, answered "32" instead of "6" when asked to substitute 2 for a in 3a , thus confirming that most of them remained in their initial arithmetic frame of reference.

The questions dealing with the meaning of the letter within the context of addition (4 + b), yielded some changes between the pre-test and the post test. For Frankie and Gail, the change was significant: in the pre-test they had transformed the expression into an equation, that is, they interpreted the letter as a "specific unknown". In the post-test, they indicated that the letter could take on more than one value " 4 + 2, 4 + 3" (generalized number). A minor change was observed in the case of Wendy who in the pre-test had used an alphabetical interpretation (4 + b = 4 + 2) and who now wrote "4 + b = 6 and 4 + n = 12". Antoinetta and Filippo showed no change from the pre-test. Antoinetta maintained her interpretation of the literal symbol as a generalized number (4 + b is 4 + 3 or 4 + 6) and Filippo still used a specific unknown interpretation for the letter by writing "4 + b = 8" . Yvette's response was too vague to draw any conclusion.

For the simplification question involving only like terms (2a + 5a), five of the six subjects maintained their original place-value interpretation such as 2a + 5a = 22 + 53 . Antoinetta was the only exception: she read the expression correctly ("2 times a plus 5 times a") but then she wrote "2a + 5a = 10" explaining "You don't know what the a is so I just times the 2 and the 5". For the simplification question involving unlike terms (3a + 4a + 5), Wendy, Filippo and Yvette maintained the place-value interpretation they had expressed in the pre-test. The other students indicated some change. Antoinetta solved the problem by simply adding the numerals (12a). Frankie first drew a rectangle whose area was to correspond to the given expression and also said that it could mean "3 x 2 + 4 x 2 + 5". Gail drew a correct line problem



but still could not simplify and wrote "3 + 4 + 5"

For the questions requesting the addition of 4 to given expressions, the responses of all subjects , excepting Antoinetta, were similar to those of the pre-test: Wendy continued to ignore the letter ("4 plus n + 5 is 9") and used her alphabetical place-value interpretation ("4 plus 3n is 4 plus 314"); Frankie and Gail still ignored the letter or used a place-value interpretation (4 + 5 , 4 + 32); Filippo continued to form equations and used a place-value interpretation (4 + N + 15 = 26, 4 plus 3n is 4 plus thirty something);

Yvette still added 4 to the expression (4 + n + 5) stating that it was incomplete and she also used a place-value interpretation (" 4 + 3n is 4 + 30") Antoinette did indicate some change in that she no longer evaluated the letter nor used a place-value interpretation. Nonetheless, she was unable to find a correct answer to the additive problems.

From these results, one can conclude that Wendy, and to some extent Filippo, operated in the post-test, in a purely arithmetical frame of reference. Frankie, Gail and Yvette used both arithmetic and algebraic frames of reference. Antoinette was the only student who did not revert back to her arithmetic framework and exhibited a significant evolution, since most of her responses were within an algebraic context. The fact that five of the six students did revert back to a framework which was essentially an arithmetic one, indicates how difficult it is to replace it by an algebraic one for the learner.

RETURN TO AN ALGEBRAIC FRAME OF REFERENCE

As pointed out earlier, the second part of the post-test was to ascertain how much our students had learned from the three lessons. However, to achieve this it was necessary to bring them back to the algebraic frame of reference established in the teaching experiment. This was done by raising the question:

"IN ALGEBRA, WHAT DOES 5b MEAN ? "

Upon the request to respond in algebra, Wendy, Gail, Filippo and Yvette, who had initially given an arithmetic interpretation for the concatenated expression, changed their response to an algebraic one and said "5 times b" . Not only did these four subjects respond differently as to the meaning of the expression 5b , but also, in substituting a numerical value for b, they no longer used a place-value interpretation, but used the intended algebraic meaning, that of multiplication: in substituting 2 for b, they did not say "fifty-two" as before, but now read it as "5 times 2". This change in response is interesting since it shows that all these subjects knew the meaning of 5b within the context of algebra, but unless specifically requested to respond within that context, they remained in an arithmetic one.

Once our students were induced to work in an algebraic frame of reference, they continued to answer accordingly for the remainder of Part II of the post-test. However, some of them needed continued reassurance that they were to respond within that context. Filippo, and often Yvette, constantly asked "in algebra?". Antoinetta and Frankie have not been mentioned here since they had both initially stated that 5b meant 5 times b without being prodded.

Since at the end of the second part of the post-test, our subjects were clearly in an algebraic frame of reference, we returned to some of the questions asked in both the pre-test and at the beginning of the post-test, expecting that some changes in their responses would occur. A change did occur, but it was not always a clearcut one. Five out of the six subjects indicated the presence of both an algebraic and an arithmetic frame of reference. Wendy had previously ignored the letter in the question "add 4 onto 3n", but now she wrote "3n + 4". However, when replacing a by the number 2 in 3a she claimed 3a could be "thirty-two or 3 times a". Frankie had previously used a place-value interpretation (4 + 32) but now claimed that 3n was "3 times a number", but he was unable to accept the lack of closure of the expression as evidenced by his replacement of the letter n by the number 4, thereby reverting to an arithmetic framework. Further indication that Frankie had abandonned his place-value interpretation can be found in his rewriting "2a + 5a" as "2 x a + 5 x a" whereas before, he had written "22 + 53". Gail now provided only algebraic answers. For "add 4 onto 3n" she now wrote " 4 + 3n " ; in simplifying "3a + 4a + 5" she wrote "3 x a + 4 x a + 5". However, in commenting her initial answers, she claimed that "2a + 5a" could be "twenty something plus fifty something in ...adding or subtracting". Thus she hinted at the possibility of an alternate answer in arithmetic. Filippo answered only within the context of algebra. For example, he wrote "4 x 3 x N" for "Add 4 onto 3n". However, before answering any question he preceded his answer by "in algebra?" Yvette responded "3n + 4" to the question "add 4 onto 3n", but the remainder of her answers were very erratic indicating a conflict between the two frames of reference. Our sixth subject, Antoinetta, was not listed above since she was the only student who had spontaneously answered the questions in Part I of the post-test within an algebraic frame of reference.

CONCLUSION

The research reported here presents strong evidence regarding the cognitive conflict created by the existence of both an arithmetic and an algebraic frame of reference in the mind of the novice algebra student. Our six subjects were taught individually, and after three lessons could generate non-trivial algebraic expressions as answers to given dot array, line segment length or rectangle area problems, and conversely, they could generate such problems for given algebraic expressions. Yet, a month later, five of our six subjects reverted back to a framework which was primarily an arithmetic one. The explicit instruction to answer in algebra led to a remarkable shift in their responses. This is most evident in the answers to the problem "add 4 onto 3n". Whereas their initial response in the post-test was essentially arithmetic (4 + 32, 4 + 314), when jolted into an algebraic frame of reference, they all answered "4 + 3n". These results imply that in early algebra, the teacher and the researcher cannot take the student's responses at face value. We must first reassure ourselves that he is well aware of the frame of reference relevant to our inquiry.

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THE EFFECTS OF COMPUTER USE ON THE ACQUISITION OF MULTIPLICATION FACTS BY A LEARNING DISABLED STUDENT

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INTRODUCTION

Although there are a number of articles concerning the use of microcomputers with special populations, there is little research being done on the effectiveness or impact of the use of microcomputers with these populations (Hoffmeister, 1982; Blaschke, 1985). This is especially true as it concerns students who are diagnosed as having learning disabilities (Hasselbring and Crossland, 1981; Shiffman, Tobin and Buchanan, 1982). It may be that the use of computers and educational software will facilitate the remediation of specific learning problems associated with various types of learning disabled students, and for the learning that takes place to remain stable over time. This study will attempt to investigate the effects of the use of computer and two types of mathematical software on the acquisition of multiplication facts by a learning disabled student in a special educational setting.

The demand for integrating computers into special education programs for the mildly handicapped is presently oriented primarily to the use of computer assisted instructional (CAI) software intervention. The primary models, or vehicles, for the delivery of instruction have been drill and practice programs, which still consititute approximately 60% of the high priority instructional courseware needs according to special education administrators. The other 40% of the courseware reflects an expressed need for more tutorial, or tutorial-based programs (Blaschke, 1985).

Research on the usefulness of drill and practice that is specifically focused on the use of mathematical concepts with learning disabled students has shown that drill and practice programs do not affect students' performance if they are using a reconstructive strategy for determining a solution to a addition problem (Hasselbring, 1985). However, this study also found that: 1) almost all of the students increased their rate of correct responding as a result of exposure to the drill and practice program; and 2) few students moved from the use of reconstructive processes (a more primitive strategy) to the use of more sophisticated reproductive processes for solving mathematical computing problems.

In view of this information, we are undertaking a series of investigations which seek to discover the relevant dimensions of the use of both software and teacher intervention strategies with mildly handicapped L.D. students within the area of mathematics. In this particular presentation, two studies are presented, a pilot study (Study #1) involving the use of drill and practice software, and a continuation study (Study #2) involving the use of tutorial-based software under varying conditions of teacher intervention. Study #1 used a single subject, multiple baseline ABAB design, while Study #2 used the same single subject with a multiple baseline withdrawal design (Tawney and Gast, 1984) in order to determine if:

1. The use of drill and practice software as an effective intervention strategy for a specific mathematics disability involving the multiplication tables.

2. The use of specialized tutorial-based software employing a "gradual recall" method (Skinner, 1974) under varying conditions of teacher intervention as an effective intervention for the acquisition of multiplication facts by a leaning disabled student.

3. The long-term effects of computer-based learning when it is directed at information meant to be committed to memory, in this case, the multiplication tables, 7 - 9.

PROCEDURES

SUBJECT: The student selected for this study was a male, 16-year old high school student with a specific learning disability in the area of mathematics. The student exhibited no innappropriate behaviors and was highly motivated to learn the multiplication tables, but had experienced years of failure in their acquisition.

METHOD: The multiple interventions for this study were sequenced as follows:

> STUDY #1: A drill and practice program providing instruction in a variety of multiplication problems, without teacher intervention, followed by a return-to-baseline condition, and the final intervention phase.
>
> STUDY #2: A tutorial-based program that provided practice using a "gradual recall" technique, without teacher intervention, followed by a return-to-baseline period. A shift was then instituted using the same tutorial-based program with specific teacher intervention in teaching a problem-solving strategy for computing multiplication problems. Finally, a series of probes will be conducted over the following three months to check on the stability of the learning over time.
>
> The student was exposed to each of the conditions successively and, upon stabilization (a minimum of three trials of criterion responding (Tawney and Gast, 1984)), were returned to a baseline condition for at least three sessions. The return-to-baseline conditions consisted of no computer intervention and evaluation of daily performance using a standard set of 24 multiplication problems involving the 7, 8, and 9 times series.

RESULTS

STUDY #1 : Figure 1 displays the number of errors across the sessions. In the first baseline condition, errors increased from 0 during session 1 -- to 4 errors during session 3. During the first intervention the drill and practice software was introduced to the student. The subject used the software for about 20 minutes a session and errors decreased to 1 by session 6. After the baseline condition was reintroduced, the error rate climbed back up to 2 by season 9. The second intervention period started off with an increase in errors. The subject decreased errors from 3 during session 10 to 0 during sesion 11. The error rate then began to climb from 1 during session 12 to an average of 2 during the final session.

FIGURE 1

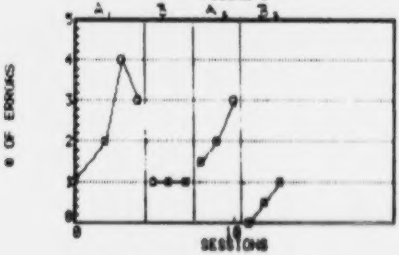
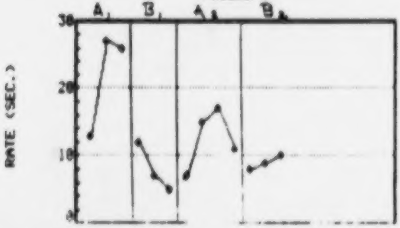


Figure 2 illustrates the average amount of time that the subject required to answer each of the 10 problems. The student's time went from a low of 14.6 seconds to a high of 26.9 seconds during the first baseline period. The first intervention period started on session 4 and the student's times for the intervention were 10.1, 6.6, and 6.1. The average time in the second baseline went from a low of 7.2 during session 7, to a high of 17 during session 9. The subject decreased his response time during the first 2 sessions of the second baseline period. The subject's time then began to increase during session to 8.3 seconds and to 10.3 seconds during the last session.

FIGURE 2



STUDY #2 : Figure 3 illustrates the results-to-date of the second study, which involve an initial baseline period, followed by the use of the tutorial package. After the baseline responding stabilized (4 days) on the timed tests, the student was exposed to the tutorial software with the number of errors slowly decreasing to an average of 1 error per 20 problems by the 10th day of intervention. The study is presently at this point and once the responding has stabilized at 1 error per 20 problems for three days, a return-to-baseline condition will be in effect during which probes will be conducted to test for the stability of the learning. If the error rate increases over time, then the teacher intervention phase of the study will be instituted and carried out unil error rates are now again at a stable rate, at which time another baseline condition will be instituted.

FIGURE 3

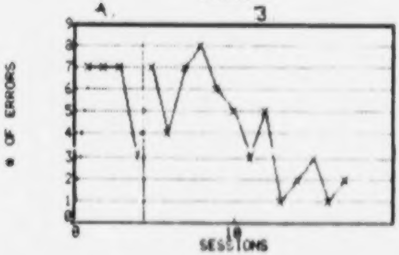
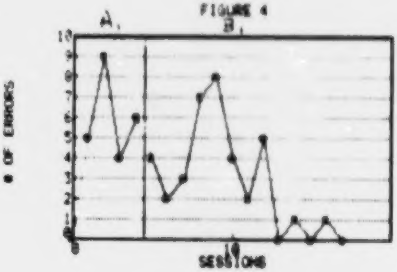


Figure 4 illustrates the student's responding under untimed testing conditions, where the student had as much time as he wanted for responding. Basline responding was generally more erractic, with a median response error rate of approximately 6 errors per 20 problems. However, the first intervention phase (tutorial software) shows a similiar pattern of responses as with the timed conditions. At the present time, error rate has begun to stabilize at 1 error per 20 questions, the pre-experimental criterion level of "acceptable" errors.

FIGURE 4



CONCLUSIONS

These findings indicate that drill and practice software may make an initial, but transitory effect upon the number of errors and the amount of time required to successfully complete multiplication problems. It appears that wihout a specific intervention treatment which seeks to change the strategy by which the student approaches the problems, any gains made during to the computer interaction will not hold over time. One of the primary limitations of the pilot study was that none of the baseline or intervention periods were long enough during the first study which may have introduced more variation in response patterns if 5-10 days were given to each period. In addition, it was found that the measure of rate of problems solved was not as sensitive a measure of behavioral change as having both timed and untimed tests of problem solution. This mode of testing is more realistic in that it allows the student the opportunity to respond under low-stress and high-stress situations that more closely resemble normal testing situations.

The directions for additional research indicated a need to use CAI software that introduced a strategy for solving multiplication problems within a tutorial-based framework, and to possibly manipulate the type of teacher intervention. It was with these considerations in mind that the second study was designed so that the student was first exposed to a tutorial software program that was designed to remediate memory-deficit problems. Provisions were also made to intervene with a specific teacher intervention in the form of a new reproductive strategy if the gains made with the tutorial software did no hold over time.

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EXTENSIVE AND INTENSIVE QUANTITIES IN MULTIPLICATION AND DIVISION WORD PROBLEMS: A PRELIMINARY REPORT AND A SOFTWARE RESPONSE

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Joel S. Poholsky, the Educational Technology Center

Two studies examined, respectively, the kinds of one step multiplication and division problems generated by students in grades 4-13, and then the kinds of problems such students found difficult. Results confirm and complement recent work by others regarding the shortcomings of student cognitive models of multiplication, division, and intensive quantities (generalized rates). Results also show a tight fit between problem types not generated by students and the problems they find difficult. We describe a coordinated multiple representation software environment under development intended to help build and coordinate the student cognitive models now lacking.

Recent work by Fischbein, Greer, Bell, and others has made plain the paucity and inflexibility of student models for multiplication, division and rate - or "intensive" - quantities, (Ekenstam, 1983; Greer & Managan, 1984; Bell, Fischbein, & Greer, 1984; Fischbein, 1985). They manipulated the numbers in word problems to violate the numerical assumptions of student primitive cognitive models. The resulting decline in performance highlights the shortcomings of the students' primitive models, and also confirms specific characteristics of those models. In particular, the primary student model of multiplication is repeated addition. Others, most notably Usiskin and M. Bell (1983) and Freudenthal (1983), have analyzed the various meanings of multiplication and division and have likewise concluded that students now receive little instruction that would build richer and more flexible cognitive models. Our approach is based on (1) the distinction between extensive, E, and intensive, I, quantities (Freudenthal, 1973; Schwartz, 1976, 1984), and (2) attention to the semantic relationships among the referents of the quantities in a given problem (Quintero, 1981, 1983).

There is a direct relationship between the two types of division and the role of intensive quantities (generalized rates) in division problems. (Given a set of size p to be subdivided, the partitive interpretation of p/q answers the question of what is the size each of q parts, while the quotative interpretation answers the question of how many parts of size q.) First note that the quotient of two extensive quantities with different units, E/E, yields an intensive quantity, I. Now, using Bell's example (1984), we can illustrate the relationship between the two types of division and quantity-types concretely.

Partitive: distance (E) / time (E) = speed (I)

Quotative: distance (E) / speed (I) = time (E)

While Bell (1984) found better performance on the partitive (E/E) problem, he showed performance to be sensitive to the interaction between the contextual and numerical features of the problem. He found that multiplication problems amenable to a repeated addition interpretation were easier than those involving the product of an extensive and an intensive quantity (ExI or IxE in our terms and a "rate" problem in theirs), and these were easier in turn than size-change problems. These results are consistent with the other work cited above.

We now describe a pair of experiments that support the general theme of the paucity and inflexibility of student cognitive models for these operations. Finally, we shall describe the outlines of a software environment under development designed to help build the appropriate cognitive models and to link students' primitive models to more powerful mathematical ones.

TWO STUDIES

Study A requested 290 public school students approximately uniformly distributed across grades 4-13, abilities, and sex to write single step multiplication or division word problems. 84% of the multiplication responses were true single step multiplication word problems, and 90% of the division responses were single step division word problems. Of the multiplication problems generated, 84% were of the IxE (rate) type, and 16% of the ExE type (all of which were area problems). Of the division problems generated, 81% were of the E/E (partitive) type, 17% were of the E/I (quotative) type, and 2% were of the I/I type.

Study B requested 255 students from grades 4-12 to write (but not execute) the arithmetic computation they would use to solve each of 11 single step word problems. Overall, the three most difficult problems at all grade levels were the I/I, the I/E, and the ExE - combinatoric. Apparently, direct use of a partitive analysis with an "I" dividend was difficult for most students, although an independent request of high ability 12th graders to picture their approach revealed a uniform use of partitive pictures, even when the quantities were not amenable to direct depiction (Kaput, 1985). Subsequent clinical work has also indicated that the combinatoric ExE case is learnable at the earlier grade levels via a tree diagram approach, but, as with the other meanings for multiplication and division, it is not in the typical student's repertoire simply because the student has not been offered significant experience with it.

The congenial numbers used throughout the two studies complement the work cited earlier in that (1) the earlier studies used less congenial numbers within the categories that our students found reasonably easy, hence exposed subtleties within those categories that our studies would not have found; and (2) we included problem types in Study B that were not represented in Study A or in the cited work, hence confirming the boundaries of existing student models. Note further that the unrepresented problems of Study A as well as the most difficult problems of Study B were those that were least amenable to simple partitive, quotative or repeated addition interpretation, likewise defining the limits of operant student models. As might be predicted, students were most able to solve those problem types for which they had ready cognitive models (as evidenced in Study A) - although the cited work shows that the cognitive models the students were employing were decidedly limited in their flexibility. Full data on these studies and a discussion of some of the semantic features of quantity referents can be found in (Schwartz, 1984).

A SOFTWARE DEVELOPMENT RESPONSE

We are currently involved in a combination of clinical work and software development intended to create a learning environment that includes multiple and coordinated visual representation of intensive quantities and operations on them. This will provide a traversable ramp from students' primitive models first to intermediate models and then to more powerful ones. Our strategy is to display how actions and consequences in one representation have counterparts in the others, thereby accomplishing two major objectives beyond the first order objective of introducing new and flexible representations. We shall make visually explicit, hence discussable and internalizable: (1) the coordination of and translations across representations; and (2) the structural mathematical commonalities present across representations (Shavelson & Salomon, 1985).

Our plans utilize four linked representations: (1) a series of concrete iconic models of intensive quantities (varying as to differing semantic features, including discreteness and continuity of number referents), (2) a table of data that records the numerical data resulting from actions in the other representations, (3) a coordinated graphical representation of an intensive quantity as the slope of a line in a labeled coordinate plane, and (4) a numerical workpad that also provides for tracking the operations on the units involved in the problem similar to The Semantic Calculator (Schwartz, 1983). The variation in the iconic representations provides multiple linkages to differing student primitive models as well as different starting points for different problems. The total environment provides several tools for attacking a problem, and its different components engage the major portions of a student's cognitive apparatus - involving concrete perceptual processing in (1), visual imagistic processing (in 1 & 3), and the formal/linguistic processing associated with the manipulation of formal expressions (in 4). The new ingredient of such an environment is the increasing computational power now becoming available in school microcomputers that makes possible seriously interfaced multiple window learning situations that support activities with no simple analogs in static media because the latter limit the coordination of representations to serial, and often clumsily executed, actions. Further detail can be found in (Kaput, 1985).

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USE OF SUBSTITUTION PROCEDURE IN LEARNING ALGEBRAIC EQUATION-SOLVING

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Learning algebra is assumed to involve shifting from an arithmetic approach toward numbers and operations to an algebraic approach. It is hypothesized that during a period of transition, parts of the old approach are discarded as new methods are found to replace them. This hypothesis was explored in a study on equation-solving. The use of the substitution procedure was found to be inadequate by six algebra novices and was gradually replaced for certain equation-types by algebraic methods.

It is assumed that the learning of algebra implies a shifting from a reliance on knowledge of numbers and arithmetic operations to a reliance on algebraic operations, rules, and numerical structure. In elementary school, children learn to add, subtract, multiply, and divide pairs of numbers. The sight of an addition sign between two numbers means simply to add the two numbers and give the answer. There is no ambiguity to the task. However, in algebra, an addition sign does not necessarily imply that one is to add the given numbers. For example, the addition sign in $2\underline{x} + 5 - \underline{x} = 27$ does not mean that one should add the 2 and the 5. Here the addition sign means something else. This gives rise to the following question. How do children entering secondary school make the transition from what we might call the arithmetic approach to numbers and operations to what we might call the algebraic approach? What are the major factors involved in this transition period?

The main feature of the arithmetic approach, as far as our argument here is concerned, is the sequential nature of the performance of operations. This means two things. First, any string of arithmetic operations is carried out in a left-to-right order (Kieran, 1979). For example, the problem "$3 + 4 - 1 + 5 = ?$" is calculated in the sequence: $3 + 4$ yields 7, $7 - 1$ yields 6, $6 + 5$ yields 11. Second, it also means that equations like $4 + \underline{x} - 2 + 5 = 10$ are attempted by trial-and-error replacement of the unknown term, each replacement being tried out according to the left-to-right sequence of operations until a value is found which yields a total of 10 (Kieran, 1981).

In equation-solving, this implies that the arithmetic approach does not include methods involving inverse operations. The given operation signs are used as they are; they are not signals to use the inverse operations. Finding the value of the unknown term of an equation by means of the arithmetic approach involves using the surface operations.

The algebraic approach, on the other hand, makes use of the structure of the number system. The relationships among the four operations, in particular the inverse relationship between addition and subtraction, multiplication and division, allow for solving methods which are unavailable to those possessing only the arithmetic approach. For example, an algebraic approach to solving the equation $4 + \underline{x} - 2 + 5 = 10$ might use the given signs as indicators for selecting the required inverse operations, e.g., $\underline{x} = 10 - 5 + 2 - 4$. Then the resulting operations are carried out as in the arithmetic approach. Thus, the algebraic approach includes the arithmetic approach. That is, it includes the ability to decide when the addition sign means that one is to add and when it means something else. But the algebraic approach involves more than the use of both surface and structural operations. Though it cannot be discussed fully here, the algebraic approach also includes systematic rules for comparing numerical expressions without having to evaluate them directly and allows for operating on literal terms as opposed to operating solely on numbers.

An issue which arises here is why anyone whose arithmetic approach has worked well for them in the past would change it for an algebraic approach. A second issue is how the change occurs. Does the learner attempt to throw out the old arithmetic methods and start afresh, or is there rather an attempt to graft some new processes and rules onto the old system? Matz (1979) suggested that algebra learners fit and stretch their existing knowledge in acquiring new knowledge. We agree that this probably forms the basis of acquiring the algebraic approach, that children beginning the study of algebra do not attempt to discard completely their old arithmetic approach and start afresh. Rather they attempt to fit the new approach which they are being taught to what they already know. This suggests that they will be receptive to instruction which makes some sense in terms of their old approach. They will adopt new methods for tasks which cannot be handled by old methods. But we take as a working hypothesis that this is only part of the picture. Just as they will keep certain parts of the old approach which work for them in certain tasks, they will discard the parts of it which do not work. Thus, this period will be characterized by a selection of parts of the new approach and rejection of parts of the old approach. In effect, during the early part of this transition, learners will not be searching for one single method which will work for all task-types. Rather they will latch onto specific methods for specific tasks. The result will be a kind of patchwork quilt of various methods, each one being effective for a particular type of task. As the structure of the system gradually becomes clearer, it is hypothesized that one of the methods of the patchwork will emerge and replace all the others. By this time, the transition period will be nearly completed.

The aim of this paper is to report on a subset of the results from a study dealing with the early part of this transition period. Specifically, we intend to look at one particular equation-solving method which is based on the arithmetic approach -- that of substitution. In discussing the evolution of the use of this procedure, we bring in both of the issues mentioned above, i.e., why the novice algebra learner changes her arithmetic approach and how she changes it.

THE STUDY

The first phase of the three-phase study involved interviews with ten seventh graders (12½ to 13½ years) who had never studied algebra before; this phase was designed to uncover some of their pre-algebraic notions, in particular those on equations and equation-solving. A subset of this pre-algebra group (six children) was retained for the second phase of the study: a three-month teaching experiment on equation-solving. The solving method emphasized was one which focused on the equivalence structure of equations and equation-solving, that of performing the same operation on both sides of the equation. The teaching experiment with these novices included a pretest interview and two posttest interviews, one in June and the second one in September after the summer break. The third phase involved interviews with nine intermediate algebra students who had all had at least one year of algebra instruction. They were from grades 8 to 11 (six from grade 8, one each from grades 9 to 11).

This last phase was included to provide us with 1) a clearer idea of the quality of the algebraic approach of more experienced students and 2) a barometer against which to compare the novices' equation-solving methods and views of equivalence structure.

The data for this paper are drawn from the following equation-solving situations. The novices were presented with several sets of equations which they were asked to solve throughout Phase 2: in the pretest, at each session of the teaching experiment, in the two posttests. Each of the equation-sets of the teaching experiment contained 12 equations of the following types (only the numbers were changed in each set): $6b = 24$, $2x - 6 = 4$, $y + 596 = 1282$, $16x - 215 = 265$, $n + 6 = 18$, $13x + 196 = 391$, $4c + 3 = 11$, $32a = 928$, $4 + x - 2 + 5 = 11 + 3 - 5$, $3a + 5 + 4a = 19$, $2 \times c + 5 = 1 \times c + 8$, $4x + 9 = 7x$. The pretest and posttests included equations of the same types, plus some extra ones (pretest: $37 - b = 18$, $30 = x + 7$; posttests: $x/4 + 22 = 182$, $25y + 13y + 76 = 380$, $12 + 15a - 7 + 6a = 4a + 107$). The intermediate subjects were asked to solve one set of equations during their Phase 3 interview. This was the same set as was presented to the novices in their Phase 2 pretest.

RESULTS

The substitution procedure consists in replacing the unknown term(s) of an equation by various numerical values until the correct one is found. For example, one uses the substitution procedure if in attempting to solve $4c + 3 = 11$, one tries, let us say, 3 as a value for c (4 times 3 is 12, plus 3 is 15), and then perhaps 2 (4 times 2 is 8, plus 3 is 11). In this case, the solution has been found after two trial values.

In the Phase 2 pretest, four of the six subjects used the substitution procedure 26 times while attempting to solve 72 equations in all. The remaining two subjects began to use substitution only after the first instructional session. In that session, the instruction had focused on the construction of equations from arithmetic identities and on the explicit left ↔ right equivalence structure of equations. However, over the course of the three-month teaching experiment, all subjects except one clearly decreased their use of the substitution procedure. They tended to retain it only for the last four equations of each equation-set (see Table 1). By the time of the June posttest, the frequency of usage of the substitution procedure had dropped to 14 out of 72. This is in comparison with the group of intermediate subjects one of whom used the substitution procedure only once on the same set of equations.

Table 1. Frequency of Use of Substitution in Each Equation-Type per Set

Equation-Set / Sample Equations	1*	2	3	4	5	6	7	8	9 ..	JP	SP
$6\underline{b} = 24$											
$2\underline{x} - 6 = 4$	2	3	3	1	1	2	2	2			
$\underline{y} + 596 = 1282$											
$16\underline{x} - 215 = 265$	2	2	1								
$\underline{n} + 6 = 18$	1	1				1					
$13\underline{x} + 196 = 391$	3	1									
$4\underline{c} + 3 = 11$		1		2	1		2				2
$32\underline{a} = 928$	2	1									
$4 + \underline{x} - 2 + 5 = 11 + 3 - 5$	4	3	2	2	3	4	1	1	4	3	3
$3\underline{a} + 5 + 4\underline{a} = 19$	4	5	5	4	4	4	4	4	2	3	3
$2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$	4	6	6	5	4	6	4	3	5	4	4
$4\underline{x} + 9 = 7\underline{x}$	4	6	6	5	5	6	6	6	4	4	4

Note. *Two of the six novices did not use Substitution at all in Set 1 (the Pretest). JP = June Posttest. SP = September Posttest.

An analysis of the equation-solving errors committed throughout the study led to some interesting findings regarding the use of the substitution procedure. Of all the errors in the study, giving-up before finding the correct trial value when using the substitution procedure was the most frequently committed error. It accounted for 20% of the novices' errors. Subjects gave up more quickly with the more complex equations. The average number of trials for the equations which "looked easy" exceeded the average number of trials for the equations which "looked difficult". Thus, because of their persistence with the less complex equations, there was a greater probability that they would find the correct solution with these equations.

Individual sessions of the teaching experiment were preceded by and also often followed by a set of equations to be solved. We had hypothesized that since subjects would be more tired at the end of a session than at the beginning, they would be less persistent and less efficient in their use of substitution as a solving procedure. But this was only half correct. They became more efficient with the equations which "looked easy" but in general less persistent with the equations which "looked difficult". Nevertheless, despite their success with substitution as a means of solving the less complex equations, they eventually abandoned substitution in favor of other procedures.

DISCUSSION

Why did the novices attempt to decrease the use of substitution, an arithmetic approach with which they were fairly successful, at least with the simpler equation-types? Substitution is very time-consuming. It also places a heavy burden on working memory. If the subject has not written down her trial value (and none did), she must keep track of it while performing the computations. If the left side is found to balance with the right, she must then remember what value proved successful in order to state the solution. If the equation has two occurrences of the same unknown term, she must remember when she reaches the second occurrence what it was that she used in the first occurrence and also remember the running-total up to that point. Furthermore, for the trials that are not successful, she must try to remember not to use those numbers again as trial values.

As seen earlier, the majority of the novices did not seem to use the substitution procedure with very much confidence in equations with two occurrences of the unknown. If success did not come within some predetermined range of number values that seemed inversely proportional to the difficulty of the equation, then the chances were that it would be left incomplete.

Thus, they attempted gradually to let go of substitution if they could find another procedure with which to replace it for specific equation-types. For the one- and two-operation equations, the novices began to rely more on the use of inversing -- a procedure not emphasized in the teaching experiment. For example, they began to solve $4\underline{c} + 3 = 11$ by subtracting 3 from 11 and then dividing 8 by 4. For the last four equations of each set, half of them were slowly moving towards either inversing or the procedure being taught -- that of performing the same operation on both sides of the equation. The choice of one over the other was found to be dependent on their initial preference at the outset of the teaching experiment for either asymmetric or symmetric solving procedures (Kieran, 1983), and was related to their view of the structure of an equation.

Therefore, a partial answer to the question, "How is the algebraic approach acquired?", can be found in the way that these novices stopped using substitution, an arithmetic approach, for certain equation-types as soon as they were comfortable with some replacement procedure. The replacement procedures, inversing or performing the same operation on both sides, are both algebraic. They rely on the relationships of inverse operations as their basis. But the latter procedure seemed to appeal more to those novices who had a strong view of an equation as an equilibrium structure, that is, left and right sides had always to be in balance. However, half of the novices were still using substitution for the more complex equations by the end of the study. Thus, the early part of this transition period can be said to be characterized by the use of both arithmetic and algebraic approaches, depending on the equation-type. Furthermore, it appears that the algebraic approaches which fit best with the learner's view of equations and equation-solving are the ones which are chosen to gradually replace the old arithmetic approaches.

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A LINGUISTIC MODEL OF ALGEBRAIC SYMBOL SKILL

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The fact that algebraic manipulations use a specified symbols system permits their interpretation as a language in the sense of Chomsky (1957). Methods of generative transformational linguistics have been adapted to the study of this language.

INTRODUCTION

Methods of generative transformational linguistics have been adapted to the study of algebraic symbol skill. In this paper, I shall:

1. Describe the linguistic methods used;
2. Sketch the linguistic model derived from those methods;
3. Discuss the psychological evaluation of a linguistic model;
4. Compare linguistic methods of psychological investigation with the dominant cognitive science approaches; and, time permitting,
5. Illustrate the formulation of psychological hypotheses from the linguistic model.

Table 1
Sample Sentences

a. $$\frac{28 - 7x^2}{6 - 5x + x^2} = \frac{7(4 - x^2)}{6 - 5x + x^2} = \frac{7(2 - x)(2 + x)}{(3 - x)(2 - x)} = \frac{7(2 + x)}{3 - x}$$

b. $$\begin{aligned} 3x - 5 &= -7x \\ 10x &= 5 \\ x &= \tfrac{1}{2} \end{aligned}$$

c. $$\frac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} = \frac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} \times \frac{\sqrt{x + 1} - 3}{\sqrt{x + 1} - 3} = \frac{x + 13 - 7\sqrt{x + 1}}{x - 8}$$

d. $$\begin{aligned} 3x + 2y &= 8 \\ 2y &= -3x + 8 \\ y &= -\tfrac{3}{2}x + 4 \end{aligned}$$

THE LINGUISTIC ENTERPRISE

Generative transformational linguistics is a formal enterprise. A *language* is idealized as "a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements" (Chomsky, 1957, p. 2). For natural languages, the elements are phonemes, however, as Chomsky acknowledges, "the set of 'sentences' of some formalized system of mathematics can be considered a language" (p. 2).

In symbolic elementary algebra, the sentences are taken to be the usual simplifications of expressions, solutions of equations, etc., as are ordinarily produced by competent manipulators of algebraic symbols. See Table 1. In the present grammar, however, only expression simplification sentences (such as a and c in Table 1) are considered. The basic elements of the grammar are the symbols =, -, +, $\sqrt{\ }$, —, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, ... x, y, z, (,), [,], {, }, as well as the spatial markers of horizontal juxtaposition, vertical juxtaposition, and diagonal juxtaposition.

For languages, not all of the possible sequences of symbols are well formed (grammatical) sentences. In the simplest terms, the formal objective of the linguistic analysis is the specification of a set of rules, called a grammar, which generates all of the sentences of the language but none of the non-sentence combinations of elements.

THE LINGUISTIC MODEL OF SYMBOLIC ALGEBRA

Each expression simplification sentence can be considered as a sequence of algebraic expressions. Consequently, the grammar is concerned first with generating individual expressions, and then with generating expressions compatible with a given expession. Each expression is presumed to have a deep form (DF) which explicitly represents the parse and the operations which may only be implicitly represented in surface form (SF). For example, the surface form $3x^2$ has a deep form 3M[xE2], where the capital letters are abbreviations for operations, and the parentheses indicate parsing in the usual way.

PHRASE STRUCTURE GRAMMAR

The first component of the grammar is a *phrase structure grammar* which generates the deep form for every possible algebraic expression. A phrase structure grammar is a series of rewrite rules which takes an initial symbol "Z" and specifies replacements for symbols until only a special class of terminal symbols remain.

The rewrite rules of the grammar are

Z --> [ZOZ]
Z --> [NZ]
Z --> V
Z --> Q

O --> A, S, M, D, E, R
V --> a, b, c, ... x, y, z
Q --> 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ., —

The symbol "O" represents "operation" and its replacements reresent the six usual operations on real numbers. "N" is interpreted as the unary operation "negation." "Q" and "V" represent "quantity" and "variable" respectively. "Q" can be replaced by any rational number constructed from the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ., — according to a presumed "grammar" of arithmetic. The arithmetic grammar is not elaborated in the present theory. A derivation is completed when all of occurences of Z, O, V and Q are eliminated.

As an illustration, the deep form of the expression $\sqrt{-16x^3}$ is generated from the phrase structure grammar by the following application of the rewrite rules:

Z --> [ZOZ] --> [ZRZ] --> [QRZ] --> [2RZ] --> [2R[NZ]] -->
[2R[N[ZOZ]]] --> [2R[N[ZMZ]]] --> [2R[N[QMZ]]] --> [2R[N[16MZ]]]
--> [2R[N[16M[ZOZ]]]] --> [2R[N[16M[ZEZ]]]] --> [2R[N[16M[VEZ]]]]
--> [2R[N[16M[xEZ]]]] --> [2R[N[16M[xEQ]]]] --> [2R[N[16M[xE3]]]]

The claim for this phrase structure component is that every possible derivation employing these productions and ending with only terminal symbols results in a valid deep form for an algebraic expression. Furthermore, the deep form for every possible algebraic expression is so derivable.[1]

TRANSLATIONS

Once equipped with the DF for every possible algebraic expression, the next component of the grammar concerns the translation of DF to SF. There are four distinct stages of translation postulated in the present theory, each of which must be completed before the next commences. The first stage involves the deletion of parentheses made redundant by a conventional heriarchy of operations. For example, in Stage 1, the DF, [2R[N[16M[xE3]]]] is reduced to 2R[N16MxE3]

The next stage involves the translation of operations to surface form. For example, in this Stage 2R[N16MxE3] becomes $^{2}\sqrt{-16x^{3}}$.

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[1] Actually, this claim is exaggerated. Obligatory transformations are needed to block certain DF's (e.g. xD0, 2R[N4], etc.), and to carry out stylistic adjustments (e.g. xM3 --> 3Mx, etc.).

Stage 3 effects the removal of parentheses made redundant by physical artifacts of the representation of operations: $\sqrt[2]{[-16x^3]}$ becomes $\sqrt[2]{-16x^3}$. Finally, Stage 4 performs certain final adjustments to SF such as the deletion of "2" in square root signs. In similar fashion, the translation component derives the SF of any expression, from its DF.

TRANSFORMATIONS

Having dealt with the generation of DF's by the phrase structure grammar, and the translation of DF to SF, the next component of the grammar generates the deep form of one expression from the deep form of another. For example, a transformation [αMβ]A[αMγ] ---> αM[βAγ] would allow the derivation of [3Mx]M[[yE2]Az] from [[3Mx]M[yE2]]A[[3Mx]Mz]. The transformations of the grammar correspond to selected properties of real numbrs.

Figure 1 displays the linguistic model of expression simplification sentences of algebra. As an example, the sentence $\frac{(2x)^2}{x} = \frac{4x^2}{x} = 4x$ is generated by the grammar as follows: The phrase structure grammar generates the DF, [[[2Mx]E2]Dx]. Translations carry this DF to the SF, $\frac{(2x)^2}{x}$. Successive transformations are applied to deep forms carrying [[2Mx]E2]Dx ---> [[2E2]M[xE2]]Dx ---> [4M[xE2]]Dx ---> [4M[xMx]]Dx ---> [[4Mx]Mx]Dx ---> [4Mx]M[xDx] --> [4Mx]M1 --> 4Mx. The dotted lines between DF's indicate that only *some* of the intermediate

Figure 1
The Linguistic Model

$$\begin{tikzcd}
 & SF_1 & & & SF_2 & & SF_3 & \cdots & SF_n \\
PSG+(OT) \arrow[r, dashed] & DF_{1,1} \arrow[u] \arrow[r, dashed] & DF_{1,2} \arrow[r, dash, dashed] & \cdots \arrow[r, dashed] & DF_{2,1} \arrow[u] \arrow[r, dash, dashed] & \cdots \arrow[r, dashed] & DF_{3,1} \arrow[u] \arrow[r, dash, dashed] & \cdots \arrow[r, dashed] & DF_{n,1} \arrow[u]
\end{tikzcd}$$

"SF" represents surface form.
"DF" represents deep form.
"PSG" stands for phrase structure grammar.
"OT" stands for obligatory transformations.

DF's are translated to SF.

SEMANTIC COMPONENT

Besides sentences such as $\frac{(2x)^2}{x} = \frac{4x^2}{x} = 4x$, the grammar can also generate sentences such as $\frac{(2x)^2}{x} = \frac{(2x + y - y)^2}{x^1} = \frac{(2x + 1y - y)^2}{4^0 x^1}$ which are syntactically correct, but seem to lack some quality of direction or meaning. A semantic component of the grammar is needed to formalize such notions as "simplify," "reduce," "rationalize," "factor," etc., in order to constrain the generation of such sentences. What is proposed is the delineation of canonical forms for these procedures, however, the semantic component of the grammar has not yet been undertaken.

PSYCHOLOGICAL EVALUATION

The grammar generates the sentences of algebra by means of a phrase structure component which produces the deep forms for expressions; transformations which map DF to DF; and translations which permit DF's to be manifest in standard algebraic notation. What kind of psychological claims can be advanced for such a model?

Clearly, a grammar is not a process model. An actual instance of algebraic simplification would presumably start off with an expression already given.[2] It would not be generated through the internal processes of the algebraist, by a phrase structure grammar or any other device. Nevertheless, in its overall shape, the grammar does characterize certain structures (deep and surface forms) which may be hypothesized to underlie mental representations. As well, it details translations and transformations which mediate between these structures. Thus, while not itself a psychological theory, a grammar, provides for many of the operations which could be included in a process model. Informally, a process model might correspond to Figure 2, which is closely related to the linguistic model. Presumably an expression given in surface form would be translated into deep form and subsequently transformed into a series of further deep

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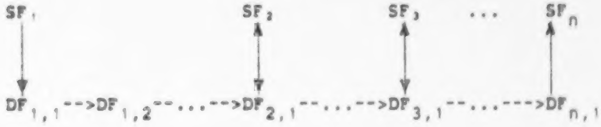
[2] This might be derived from translation of a "real world" situation; represent an application of a scientific formula; have arisen in a calculus computation; or simply been presented in an algebra text.

Figure 2
Informal Process Model



"SF" represents surface form.
"DF" represents deep form.

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forms with occasional translation for transcriptive purposes into surface form. The double arrows between DF and SF indicate that in the process model it is necessary to account for the decoding as well as the encoding of SF's. (In the present grammar, the translation component has been devised so that it functions bidirectionally.)

The overall veracity of the model may be assessed by ordinary psychological research techniques involving conjectures related to processing. Psycholinguistics, however, offers a variety of techniques for selection of one grammar from amongst competing alternatives (competative argumentation in the terminology of VanLehn, Brown and Greeno (1984)) when more than a single grammar is proposed. These forms of evaluation prove to be fruitful when educationally related alternatives are proposed.

LINGUISTIC VERSUS COGNITIVE SCIENCE MODELLING

As a means of exploring psychological phenomena, a linguistic approach differs markedly from the dominant cognitive science paradigm. First of all, a formal linugistic approach can only be adopted for the study of a "language," i.e. a set of sentences comprised of specified symbols. Thus the linguistic paradigm does not obtain for the study of word problems, translation from natural language to mathematical language, and a host of other phenomena of special interest to mathematics educators.[3] In contrast, cognitive science methods have been applied to a wide range of psychological domains, and indeed, to reasoning and problem solving functions *per se*. Nevertheless, within the restricted domain of algebraic symbol skill, a linguistic approach offers certain significant advantages.

A grammar is a formal entity. It can be evaluated logically as to its fulfilment of formal objectives (namely the generating of all and only the sentences of the language). Thus, before psychological issues are raised, a grammar has stood the test of logical consistency within its entire domain of application.

Cognitive scientists frequently use methods of artificial intelligence or computer simulation to guarantee the logical consistency of a theory. In practise, however, the implementation of the theory in a program is often of little assistence in psychological evaluation. As Davis and McKnight, 1979 observe, "the general tendency of simulation-writing is to be pushed toward dealing with minute details - and frequently details that are more characteristic of computers than they are of human thought" (p. 31). Thus cognitive science rarely achieves a truly formal character in psychological investigation.

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[3] If an algebraic language is identified which lists phonemes or words amongst its basic elements, then the grammar of the language must, in effect, subsume an entire natural language theory.

Most importantly, however, a linguistic model describes cognitive functioning at a level which may be more immediately useful for educational application. Cognitive science is concerned directly with the architecture of the mind. The basic elements of computational theory are taken to be metaphors for psychological processes (Davis & McKnight, 1979). In contrast, Chomsky (1965) observes:

> A generative grammar is not a model for a speaker or a hearer. It attempts to characterize in the most neutral possible terms the knowledge of the language that provides the basis for actual use of language by a speaker-hearer. (p. 9)

It is the knowledge underlying algebraic skill rather than the mental processes whereby that knowledge is manipulated which would seem to have the most chance of informing educational practice.

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Defining Mastery Orientation/Learned Helplessness in Mathematics from Students' Attributions for Success and Failure

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The connection between attributions and achievement in mathematics is dependent upon how mastery orientation/learned helplessness (MO/LH) is defined from students' attributions. In this study of 124 alegebra students it was found that using a theoretically based formulation of MO/LH ([SA+SE+FE]-[FA+ST+SO]) resulted in statistically significant correlations between MO/LH and achievement for females. A simplified formula (MO/LH = [SA-ST]) gave stronger correlations for females while a formula based only on help from others (MO/LH = -[SO+FO]) gave statistically significant correlations between MO/LH and achievement for males.

In recent years, a number of studies have investigated what students perceive to be the reasons for their successes and failures in mathematics (Dweck, 1975; Parsons, Meece, Adler & Kaczala, 1982; Pedro, Wolleat, Fennema & Becker, 1981). These reasons, usually referred to as attributions for success or failure, are believed to be associated with achievement in mathematics (Kloosterman, 1984; Reyes, 1984). The key to the association between attributions and achievement is knowing how attributions effect motivation which in turn effects achievement. In discussions of attributions and achievement, the terms "learned helplessness" and "mastery orientation" have often been used to classify individuals based on the type of attributions they make (Dweck & Goetz, 1978). Learned helpless students blame their failures on lack of ability and believe that effort has little to do with success or failure in school. In contrast, mastery orientated students are confident of their ability and believe that effort will improve performance and thus be rewarded in school. However, few students are totally mastery oriented or totally learned helpless and thus I prefer to think of mastery orientation and learned helplessness as endpoints of a mastery orientation/learned helplessness (MO/LH) continuum. As the connection between a student's attributions for success and failure and his or her achievement in mathematics is dependent upon how MO/LH is defined from attributions, it will be the purpose of this paper to explore both theoretical and empirical methods of defining MO/LH in mathematics from student scores on attribution scales.

THEORETICAL FORMULATIONS OF MO/LH

While there are a number of ways of classifying attributions for success and failure, classification of attributions in academic settings has most often been based on Weiner's (1974) categories of ability, effort, task difficulty, and luck. Some authors have focused on the ability dimension as the one of primary importance (Blumenfeld, Pintrich, Meece & Wessels, 1982) while others have focused more on the effort dimension (Covington & Omelich, 1979). Fennema and Peterson (1984) defined an MO/LH formula as follows:

$$MO/LH=[SA+SE+FE]-[FA+ST+SO]$$

(SA = success due to ability, SE = success due to effort, FE = failure due to lack of effort, FA = failure due to lack of ability, ST = success due to ease of the task, and SO = success due to unusual help from others or luck.) In brief, the Fennema and Peterson (1984) formula implies, as suggested above, that students who feel they have ability and that effort makes a difference will be more mastery oriented (and thus less learned helpless) than students who feel they lack ability and that their successes are the result of an easy task or unexpected help from others. Table 1 summarizes the attributions mastery oriented and learned helpless students are expected to make along with an explanation of how those attributions lead to steady or increased effort for mastery oriented students but decreased effort for learned helpless students.

Table 1

Attributions, Expectation of Success, and Effort for Mastery Oriented and Learned Helpless Students

Attribution	Expectation of Success	Effort on Similar Task
	MASTERY ORIENTED STUDENTS	
SUCCESS attributed to:		
1. Sufficient Ability	Expectation of success on similar tasks	Continued effort
2. Sufficient Effort	Expectation of success on similar tasks	Continued effort
FAILURE attributed to:		
1. Lack of Effort	Expectation that increased effort will lead to success	Increased effort
	LEARNED HELPLESS STUDENTS	
SUCCESS attributed to:		
1. Ease of Task	No reason to expect success on tasks of reasonable difficulty	No reason to put forth effort
2. Help from Others	No reason to expect help and thus no reason to expect success	No reason to put forth effort
FAILURE attributed to:		
1. Lack of Ability	No reason to expect success on similar task	No reason to put forth effort

EMPIRICAL FORMULATIONS OF MO/LH

Design and Instrumentation. To test the relationship between attributions, MO/LH, and achievement in mathematics, 124 ninth grade algebra students were given an achievement measure and an attribution measure. The achievement measure was the *Mathematics Basic Concepts* subtest (Level I, Form X) of the *STEP Basic Assessment Tests* (1979). Students answer using a multiple choice format, receiving an overall score between 0 and 50. Scores were also broken down by cognitive level. Of the 50 items, 30 were classified as low level (knowledge or skills); 18 were classified as high level (understanding or application); and 2 items were not classified. Split-half reliability of the scale was reported as .91 for ninth grade students (*STEP Basic Assessment*, 1979).

Students' attributions were measured by the *ALB Mathematics Attribution Scale* (Fennema & Peterson, 1984). This attribution scale consists of 8 subscales with five Likert-type items per subscale. The success subscales measure success perceived to be the result of: (1) ability, (2) effort, (3) ease of task, and (4) help from others. The failure subscales measure failure perceived to be the result of (1) lack of ability, (2) lack of effort, (3) difficulty of the task, and (4) lack of help from others. For example, the first success due to ability item was "When you figure out how to do a thought problem, is it because you are smart?". All items contained the phrase "thought problem" which was explained to subjects as one which required the development of a strategy before it could be answered. K-R 20 reliabilities for each of the subscales were calculated as part of this study (SA=.86; SE=.81; ST=.76; SO=.78; FA=.82; FE=.89; FT=.73; FO=.82).

Table 2 shows correlations between achievement and attributions for each of the subscales of the *ALB Mathematics Attribution Scale*. Because there is evidence that males and females attribute their successes and failures in mathematics differently (Pedro et al., 1981) all analyses were done separately for females and males. While only a few of the correlations were statistically significant, those that were significant were in the direction expected. For females, there was a significant positive correlation between success perceived to be due to ability and overall achievement (r=.29) and a significant negative correlation between success due to ease of the task and overall achievement (r=-.23). For males, there were significant negative correlations between overall achievement and success (r=-.21) or failure (r=-.22) due to help from others.

Table 2 also contains three definitions of MO/LH and their correlations with achievement. Using the Fennema-Peterson (1984) definition of MO/LH ([SA+SE+FE] - [FA+ST+SO]), significant correlations between MO/LH and achievement were found for females regardless of whether high level achievement (r=.21), low level achievement (r=.31), or overall achievement (r=.29) was considered. When the second definition of MO/LH ([SA-ST]) was used, higher correlations with achievement for females were found (r=.30 for high level, r=.39 for low level and overall achievement) than had been the case with the first definition. Using the third definition of MO/LH (-[SO+FO]) gave significant correlations with high level achievement (r=.22) and overall achievement (r=.25) for males. Factor analysis was used to see if additional combinations of the attribution subscales could be found which resulted in MO/LH formulations which had stronger relationships with achievement. No additional MO/LH formulations were found.

Table 2

Correlations Between Attribution Subscale Scores, MO/LH, and Achievement

	Females[a]			Males[b]		
	STEP Mathematics Achievement					
Attribution Variables	Overall	High Level	Low Level	Overall	High Level	Low Level
Success Ability	.27*	.19	.29*	.12	.13	.10
Success Effort	.03	.04	.01	-.10	-.11	-.06
Success Task	-.23*	-.20*	-.21	.02	.01	.04
Success Others	-.14	-.17	-.11	-.21*	-.19	-.14
Failure Ability	-.17	-.12	-.19	-.05	.01	-.04
Failure Effort	-.08	-.15	.00	-.07	-.13	-.02
Failure Task	.15	.20	.08	-.06	-.07	-.01
Failure Others	.03	.01	.03	-.22*	-.18	-.18
MO/LH(1) = [SA+SE+FE]-[FA-ST-SO]	.29*	.21*	.31*	.06	.02	.05
MO/LH(2) = [SA-ST]	.39*	.30*	.39*	.07	.08	.05
MO/LH (3) = -[SO+FO]	.06	.08	.04	.25*	.22*	.18

*p<.05 [a]n=61 [b]n=63

DISCUSSION

MO/LH(1). The significant correlations between MO/LH(1) and achievement for females support the definition of MO/LH from the psychological literature as outlined in Table 1. The fact that the definition appears unrelated to achievement for males is rather surprising. While there is literature to support the possibility that MO/LH may have more effect on achievement for females than for males (Dweck, Davidson, Nelson & Enna, 1978; Parsons et al., 1982), there is nothing in the literature to suggest that MO/LH is not a useful construct for males.

MO/LH(2). The second definition of MO/LH gave stronger correlations with achievement for females than the first. While MO/LH(1) follows closely from the psychological literature, the strong correlations for MO/LH(2) indicate that success effort, success others, failure effort, and failure ability attributions are not as important as expected. One possible explanation to account for the lack of importance of effort is that in algebra more so than in other subject areas, effort may be perceived to be important by all students. Thus effort would not differentiate mastery oriented and learned helpless students in algebra as well as it would in computational mathematics.

MO/LH(3). The third definition of MO/LH was formulated to try to explain something about how attributions mediate achievement for males. The fact that this definition includes only the effect of others on achievement indicates that those males who do not see help or lack of help from others as factors in their successes and failures are the ones who have higher achievement.

High vs. Low Cognitive Level Mathematics. Given the current massive push for teaching problem solving in mathematics, an attempt was made to identify the extent to which attributions and thus MO/LH affected high as opposed to low cognitive level mathematics. As can be seen from Table 2, the correlations between attributions and achievement were generally strongest in the case of low cognitive level mathematics. This is somewhat surprising given that the ALB attribution instrument spoke specifically of "thought" (high cognitive level) problems. One possible explanation for this is that students were so used to computational mathematics that even the presence of the term thought problem in the attribution items was not enough to make them reflect on high as opposed to low level mathematics when filling out the attribution instrument. Another possibility for the smaller correlations with high level achievement is that there was more variation and thus less consistency in the high level problems on the standardized achievement test than among the low level items. This would lead to greater measurement error which would effect the size of the correlations found. The fact that there were fewer high level than low level items (30 low level and 18 high level) may also have affected the accuracy of the high level scores. In short, factors other than cognitive level of mathematics could have accounted for the differences in correlations between high and low level mathematics achievement and MO/LH. Thus, conclusions about differences in the MO/LH and achievement relationship based on cognitive level must be taken cautiously.

CONCLUSION

While the three definitions of MO/LH proposed from this study are somewhat diverse, all do agree with some aspect of the literature. Definition 1 follows closely from the psychological literature as outlined in Table 1. Definition 2 reduces the importance of effort in comparison to ability and task as factors influencing achievement. This is possible given that attributions for ability have, at times, been offered as the key to achievement (Blumenfeld et al., 1982). Definition 3 shows that perceptions of the importance of help from others is more of a factor in achievement for males that for females. This agrees with sex-related difference literature (Fennema & Peterson, in press) which indicates that boys may be more independent learners of mathematics than girls.

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THE CONSTRUCTION OF A MULTI-FACTOR TEST FOR TEACHING STRATEGIES RESEARCH IN MATHEMATICS

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In teaching strategies research, it is hypothesized that different instructional treatments of a content topic result in different patterns of learned performances. To investigate this hypothesis, tests are needed that identify and measure a variety of student outcomes. Several iterations of item construction and try-out procedures resulted in the production of a thirty-six item test that assesses up to nine facets of the topic of slope and equations of a line.

The purpose of this research was to develop a prototype test to be used to compare teaching strategies based upon the kind of learned performance they produce.

The work of Anderson, J.R. (1976), Gagne and White (1978), Paivio (1971), Ponte (1982), and Tulving (1983) indicates that learners encode information and represent it to themselves in four types of memory structures: procedural memory, propositional memory, visual memory, and episodic memory. Procedural memory stores "know how"; the application of rules and the carrying out of routines in a habitual, often unthinking way. Capabilities in propositional memory give rise to "knowledge stating" behavior and the preservation of the meanings of verbal statements and symbol systems. Visual memory stores images that are analogical representations of concrete things and configurations that are encountered. Episodic memory retains personal experiences and events in relation to time sequences and is autobiographical in nature.

When teaching strategies differ in their treatment of the learner's interaction with mathematical content, different patterns of learned performances should result. As the learner engages in learning task, the kinds of encoding utilized and the forms of memory representations should be influenced by the instructional strategy. The knowledge that gets stored and where it is stored determines the pattern of acquired capabilities the learner comes to possess. To compare strategies on this basis, a test is needed that is sensitive enough to measure different performance outcomes resulting from the various structures.

The test to be developed had to a) contain factors that could be related to the performances associated with the various memory structures, b) have questions within the factors that had high internal consistency but low correlations with all other questions in other factors, and c) establish that the factors are distinct by showing that the pair-wise correlations between factors are small. A test plan was developed that identified characteristic performances associated with propositional,procedural , and visual memory structures. Within each of these representations, three levels of questions were defined: knowledge, technique and manipulation, and comprehension. This test plan resulted in nine categories and these became the nine factors that were hypothesized.

Test questions were generated over the topic of a slope of a line. After several iterations in pilot studies, a final version of a test was constructed containing 36 items, four for each factor. A panel of five mathematics educators evaluated each question and chose the type of memory and the level of understanding the item represented. The panel agreed 82% on the memory type, 77% on the taxonomic level, and 64% on both level and type.

The final administration of the test was to 395 high school and college students in twelve classes enrolled in Algebra II or equivalent. Generally, correlations of questions with the factors to which they are hypothesized to belong are high. The difference between a question's correlation with its factor and the next highest correlation with another factor is approximately .35. Correlations among the nine factors are generally low with 28 of the 36 correlations between factors being less than .30. Correlations between dimensions that represent different memory structures ranged from .50 to .55 while correlations between questions that represent different taxonomic levels ranged from .44 to .54.

The low interfactor correlations suggest that the test measures distinct dimensions of the mathematical topic of slope. The test appears to be a good first approximation to a testing instrument that can measure sensitive differences in learning outcomes and establish effects that can be attributed to a particular instructional treatment of a mathematical topic.

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YOUNG CHILDREN'S ERROR PATTERNS ON ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION WORD PROBLEMS

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Abstract

The present analysis is part of an investigation of first- through third-grade children's acquisition of multiplication and division concepts and processes. As part of this study errors were coded on six addition and subtraction word problems and six multiplication and division word problems. Using the wrong operation and stating, as an answer, a number given in the problem were the most common errors. Individual error patterns indicated groups of children who responded systematically with Given Numbers, children who added on all problems, children who seemed to add when in doubt, and children who used only addition and subtraction but who did not seem to systematically apply those operations to multiplication and division in ways expected.

Research on addition and subtraction word problems has resulted in systematic classification structures for problems, detailed descriptions of solution strategies and their relationship to the semantic structure of the problems and classification systems and models for the development of the solution strategies (Carpenter & Moser, 1983; Nesher, 1982; Nesher, Greeno & Riley, 1982; Riley, Greeno & Heller, 1983; Briars & Larkin, 1982). A subsequent and increasingly important concern is that of generating the parallel results for multiplication and division word problems and investigating the commonalities of the character and development of strategies across content domains. As part of such a larger study, error strategies across addition, subtraction, multiplication and division were investigated.

METHOD

Subjects. The subjects were 43 first-grade, 35 second-grade and 50 third-grade children in one school in a predominantly white Midwest community of 15,000 people. The sample included two classrooms at each grade level.

Procedure. In an individual interview, each subject was read and asked to solve twelve one-step word problems: 2 addition, 4 subtraction, 2 multiplication and 4 division problems. Three types of physical materials were available for use in solving -- bowls, counters, and sticks. Children's responses were coded and tape-recorded. Data collected included physical materials used, correctness of response, strategy or explanation used, and type of error if answer was incorrect.

RESULTS AND DISCUSSION

The types of errors noted were: a) Miscount-- the child used an appropriate strategy, but miscounted in some way; b) Forgets -- the child used an appropriate strategy but forgot a number and substituted a different one; c) Given Number -- the child responded that the answer was one of the numbers given in the problem; d) Wrong Operation -- the child used an incorrect operation; e) Guess -- the child incorrectly guessed at answer or used an incorrect number fact; f) Other -- the child made some other identifiable error such as giving "one" as an answer, making unequal groups for a division problem and forgetting to make the groups equal, or using the number of sets as the number of elements in a set; and g) unknown or uncodable.

The percentages of addition, subtraction, multiplication and division problems on which the error strategies were used are given in Table 1, along with the respective percentage out of the total errors on that type of problem. On addition and subtraction problems, the most frequent errors for first- and third-grade children were Given Number and Wrong Operation. For second-grade children, Given Number was still the most frequent, but Miscount occurred

Table 1

Percentage of Error Strategy Use on Each Problem Type

Error Strategy	Grade Level	Problem Type							
		Addition		Subtraction		Multiplication		Division	
		% use	% of err	% use	% of err	% use	% of err	% use	% of err
Given Number	1	31	55	17	35	20	22	31	38
	2	20	64	12	37	17	32	24	37
	3	9	35	5	26	5	12	11	25
Wrong Operation	1	8	14	16	32	40	44	33	39
	2	1	4	6	17	24	45	15	23
	3	14	54	7	41	26	63	19	44
Guess	1	3	6	4	8	7	8	5	6
	2	3	14	6	17	1	3	3	4
	3	1	4	4	21	2	5	3	6
Miscount	1	10	18	9	19	13	14	4	5
	2	4	14	8	24	4	8	6	10
	3	2	8	2	9	3	7	2	5
Forgets	1	3	6	0	0	5	5	0	0
	2	0	0	1	4	0	0	4	6
	3	0	0	0	0	1	2	1	2
Other	1	0	0	0	0	2	3	3	4
	2	0	0	0	0	0	0	4	6
	3	0	0	0	0	0	0	3	7
Unknown or Uncodable	1	0	0	3	6	3	4	6	7
	2	1	4	0	0	7	13	9	14
	3	0	0	1	3	4	10	5	11

more often than Wrong Operation. The two error strategies most used by all three grades on multiplication and division word problems were Given Number and Wrong Operation.

With Given Number errors children fell into several distinct groups. First, there were children who, when asked why they responded with a given number, said, "because you said it in the story." These children seemed not to understand the mathematical context of the story. A second group of children, who responeded with a Given Number on the Compare forms of the addition and subtraction problems (problems involving a statement or question of one person having more than another), interpreted the problem in a mathematical way -- as a comparison. These children could explain that there was a comparison, but they interpreted a "how many more" question as "which number is more" or, as one child explained, ". . . you asked how many is the more." Thus, as Hudson (1980) found, the wording of the problem did not make clear the mathematical structure of the problem.

For multiplication and division problems, a group of children who responded with a Given Number explained their answer in terms of a one-to-one correspondence. If the problem involved placing 6 marshmallows in each of 5 cups, the child answered "You need 5 marshmallows in all because there are 5 cups--so one marshmallow for each cup." This seemed even more prevalent on division problems. Finally, there was a group of children who, for division problems, would model the problem reversing the roles of the numbers. With 24 carrots to be put equally with 3 apples, they would group by 3's, but would correctly remember that the question was how many carrots with one apple. Thus, they gave "3" as answer. This error is similar to one children make with fractions, which is interpreting "thirds" as "group by 3's." (Hunting, 1985)

The Wrong Operation strategy was the most used strategy, appropriate or inappropriate, for solving multiplication and division problems, and also accounted for about 9% of the answers on addition and subtraction problems.

The majority of Wrong Operation errors were in incorrectly chosing addition. The distribution of Wrong Operation errors is shown in Table 2.

Table 2

Distribution of Wrong Operation Errors

	Grade Level		
Wrong Operation	1	2	3
Addition	37%	13%	20%
Subtraction	10%	3%	11%
Multiplication	3%	1%	4%
Division	0%	0%	1%

First- and second-grade children made more wrong operation errors, proportionally, on subtraction than on addition, whereas, third-grade children did the reverse. Children at all grade levels made more Wrong Operation errors, proportionally, on multiplication problems than on division problems. For individual error patterns, there was a group of children, mostly first grade, who viewed the story problems as a world of all addition problems. A second group appeared to think, "when in doubt, add." These results have implications for previous studies. If many children have a general, addition-dominant approach to solving word problems, their correct performance on addition word problems cannot be solely attributed to understanding the word problems. How much of the correct performance on previous studies is a result of fortuitous general strategies rather than specific understanding of the concepts in the word problems? Then, too, what underlies the formation of the general strategies, and does this have implications for instruction? Are the general strategies low-level responses born of confusion or are they high-level attempts by children to build structure where none has been provided? Implications for future studies include a heightened emphasis on determining why a child choses a strategy. Current interview techniques and problems are not sufficient for detecting the "whys," especially with less verbally inclined or less critically objective children.

For the second-grade children and, more so, the third-grade children, there is a group who use a mixture of addition and subtraction strategies. However, their inappropriate use of addition and subtraction on multiplication and division problems is not what may be expected. On multiplication problems, of the seventy-seven Wrong Operation errors 90% were addition and 10% were subtraction. One might expect, then, a large majority of the division errors to be subtraction. However, of the 114 Wrong Operation responses, 60% were addition, 32% were subtraction, and 8% were multiplication. This distrubution, as well as the large number of Wrong Operation errors brings into question the assumption that repeated addition and repeated subtraction are the "natural" interpretations of multiplication and division.

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AUTHORING LANGUAGES, A CASE STUDY

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This paper will briefly describe a variety of different kinds of authoring languages that have been used at WICAT to develop whole courses in mathematics and science. Comparisons will be made to courseware development without the use of authoring languages. Pros and Cons will be discussed, and research possibilities will be sited that are particularly relevant to psychological investigations of mathematics learning and instruction. Particular attention will be given to a project, currently midway in development, called STATISTICS BY EXAMPLE: BUILDING YOUR OWN COMPUTATIONAL PROCEDURES.

Because the instructional development utilities discussed have been linked to several types of programming languages (e.g., LISP, Forth, C, Prolog, Pascal), several programming related issues will be considered.

Computer utilities being developed for the STATISTICS course include a "spread sheet" style data base, overlaid graphing and computational capabilities which students can use to construct, modify, refine and adapt their own statistical "number crunching" programs.

In this statistics course, computational routines are treated as "models," or useful oversimplifications of reality. Non-answer-giving phases of problem solving are emphasized, including problem formulation, trial solution evaluation, the quantification of qualitative information, the examination of underlying assumptions and sources of error, and the organization, filtering, and representation of information - i.e., phases of problem solving that are the most important to people in business, law, or other professions where intelligent decision making frequently involves statistics, but seldom requires computational proficiency.

Students are treated as future *users* of statistics, not as future *doers* of statistics (i.e., those few individuals, largely residing at computer centers, who actually carry out complex statistical computations). However, students are actively engaged in the "model building" process: they build their own computatonal procedures, and examine how changes in the "model" influence the result produced. The goal is to develop "first hand" experience about how complex procedures are assembled from understandable simpler pieces, and to become comfortable at criticising, modifying, and adapting models to suit concrete needs.

Rather than beginning topics with "pre-fabricated" principles or procedures, followed by a few (usually artficial) "applications" designed to minimize computational difficulties, students *begin* units by considering realistic problem solving situation with realistic data. Then, they build, refine, modify, and adapt their own computational procedures in a manner similar to the way children build geometry procedures using LOGO programming techniques, i.e. by trying to accomplish concrete goals in mathematically rich example situations.

With computer-driven "conceptual amplifiers" (like the "symbol-manipulator function-plotter" utility that will be described in this paper, or even familiar tools like VisiCalc), problem solving in the presence of such amplifiers is becoming as important in science and mathematics as that in their absence. The problem solver no longer can be assumed to be a person working alone with only a pencil and paper for tools. Consequently, assumptions based on such an "amplified problem solving organism" may have to be considerably different from those common in past cognitive science studies. Distinctions between instruction and assessment also will become blurred as detailed instructional paths can be documented, and as profiles of both learning and forgetting can be produced.

TEACHER-CLINICIANS' EXPERIENCE IN MATHEMATICAL PROBLEM-SOLVING SESSIONS WITH INDIVIDUAL CHILDREN: IMPLICATIONS FOR THE DEVELOPMENT OF TEACHERS

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ABSTRACT

An earlier model for the development of teachers based on the teacher as learner, as observer, and as philosopher is extended here to the teacher as clinician with data developed in a year-long project studying the heuristics employed by academically talented children of grades 4-6 in problem solving. The observations of clinicians' interview behavior form the basis for the expanded model. The implications from clinicians' preparation and actual clinical conduct will be discussed as will the development of teachers generally in the expectation that they will encourage children's construction of solutions to problems leading to understanding of mathematical concepts and processes.

BACKGROUND

A model for the preparation of teachers of mathematics at the elementary level was developed and a portion of it tested with a group of teachers in the summer of 1984. The results of the pilot study were reported at the American Educational Research Association Conference by Maher and Alston (1985). In summer 1985, the further study was conducted and the results are now being reviewed. The present paper is intended as an extension of the earlier one and its elaboration is based upon the preliminary observations of teachers who were preparing to conduct structured clinical interviews with children for a study involving children's use of the heuristic process, "think of a simpler problem (TSP)" as they were engaged in solving the problem, "What is the remainder when two to the fiftieth power is divided by 3?" A description of the script for the structured clinical interview is contained in a paper by Goldin (1985).

PURPOSE

An extension of a model for the development of mathematics teachers is suggested. Observations of four teacher-clinicians conducting interviews with academically talented children from grades four through six provide the preliminary data for this report.

PROCEDURES

Approximately twelve clinicians, working in pairs, participated in the problem solving study. Because of the diversity of experience among them, teams were organized so that a more experienced clinician could be paired with a novice. Alternately, one team member was responsible for the conduct of the interview and the other served as observer and recorder. On average, the interviews lasted fifty-minutes with none longer than an hour.

Data for this report come from the observations of interviews conducted by four clinicians, two male and two female. Their teaching experience varied from less than one-year to fifteen-years and ranged from elementary to high school mathematics teaching. Transcripts of audio-tapes and records of the specific responses to questions of the protocol were used.

The Interview

For purposes of this report the interview will be divided into five sections each representing a different aspect of the problem. Section 1 involves assessing a child's knowledge of the prerequisite skills with exponents and remainders and provides for reviewing and/or introducing the concepts. Section 2 presents the problem "What is the remainder when two to the fiftieth power is divided by three?" Here the clinician is instructed to encourage the child to talk aloud and allow the child to work freely without intervention until he/she gives up, guesses, discontinues justifying guesses, is satisfied with the response, or works for 10 minutes without apparent progress. No conceptual misunderstandings or misapplications of arithmetic rules were to be corrected at this time. Section 3 directs the clinician to offer a graduated series of hints that lead to the heuristic process TSP, if the child had not already done so. Section 4 provides specific simpler problems for guiding the child through the heuristic TSP. The clinician was instructed to provide guidance only if the child, encouraged to work freely, did not detect a pattern. Section 5 provides two equivalent problems with different exponents and a third problem, "What is the remainder when three to the fiftieth power is divided by four?" In the latter case, the child should be asked to describe how to go about solving the problem and specifically instructed not to work it out.

RESULTS

Results will be given according to the five sections outlined below.

Section 1, Prequisite Skills

Some deviations in language from the script occurred in this section. For example, Clinician D (C/D) asked: "Can you do a more difficult one?" Also, in presenting the problem "What is the remainder when 17 is divided by 5?", C/A began with "Now I have an easy question for you." C/J omitted the introductory definition of exponent and moved immediately to the language of "factor".

Section 2, Problem Presentation

Clinicians experienced difficulty restraining the impulse to interrupt or lead the child in this section, contrary to the instructions in the script.

C/D stated: "And now we come to the height of the interview." The problem was then presented and after 20 seconds, C/D continued: "Can you tell me what you're thinking now?" The child responded: "Trying to figure out what two to the fiftieth power is." After 30 seconds, C/D interrupted: "Do you think it might be possible that there might be a way to do the problem without multiplying 2 to the fiftieth power? Can you try answering 2^5 divided by 3?"

In another interview, C/D stated: "Now I have a problem for you." The problem was presented and the child began to speak and was interrupted by C/D: "Maybe you should think about it and after you've thought about it, then talk." The child immediately responded: "Two to the fiftieth power is 100; 3 into 100 is 90; 100 minus 90 is 10 and that's the remainder." C/D replied: "Can you try to think of a problem that's similar to this? One that's easier to solve but looks..." Child: "Like this?" C/D: "Close to that one." The clinician waited 30 seconds and interrupted: "How about by trying a power that's less than 50? Two to a smaller power..."

In an interview by C/J, he asked: "What did you do for two to the fiftieth power?..." Child: "Instead of taking a whole page of writing all twos, I used 50. I sort of like changed it. Sort of like opposite and I could work it, but faster than would be working it out; faster than to have to write 2, 2, 2 keep going." C/J: "All right. Do you remember what 3 to the fourth power was?" Child: "Yes." C/J: "And what was that?" Child: "Three to the fourth power is when you have to use 3 four times. That's 3 times 3 times 3 times 3." C/J: "So what do you think 2 to the 50th power would be?" Child: "Using either 2, 50 times or you can use 50 writing like that 2 times." C/J: "Do you think that would be the same?"

In C/A's interview, she stated: "Now I have a problem for you, sort of a longer problem." The clinician presented the problem and the child responded that the remainder was 1 because 2 to the fiftieth power was 100 and divided by 3 was 1. The clinician continued: "What was 2 to the third power?" Child: "Six." C/A: "Can you tell me again what it means?" Child: "OK, eight; because it's 2 times three." C/A: "So it makes it sort of a different problem. So does that mean the remainder might not be that anymore? You've written down 2 times 2 equals 4." Child: "And next is 8, 16, ... It's going to take me forever to do it this way." Immediately C/A responded: "Do you think there might be someway without doing it?"

In C/P's interview, providing less than one minute for the child to work out the problem on her own, she offered: "Can you think of a problem like this only easier to solve?" The child suggested 1 to the 50th power. C/P directed the child to change the base to two and then suggested a problem "with a little number smaller than 50..."

Section 3, TSP Suggested and Section 4, TSP Guided

For C/D, section 4 followed without opportunity for the child to think of a simpler problem. After presenting the simpler problem 2 to the second power divided by 3, he readily offered the next. C/D: "Can you try a different problem, let's say, 2 to the 3rd divided by 3?" Similarly for his other interview.

For C/A: "Can you think of a problem in terms of finding the remainder for a simpler problem? What about 2 to the second divided by 3?"

In C/J's interview, there is some deviation from the script in permitting the child to think of a simpler problem. C/J: "Can you think of a simpler problem that would give you that answer? It's like this one, only simpler." In his second interview he moves to the second level question directing the child to a "simpler problem like this one" rather than a "simpler problem".

In C/P's interview, specific simpler problems were presented to the child.

Section 5, Assessment of Understanding

Once the child was guided by the clinician to detect the pattern through the generation of a series of simpler problems, a correct response was given for the two equivalent problems with base 2 and divisor 3, but different exponents. However, when the problem "3 to the fiftieth power divided by 4" was presented, either the children began by raising 3 to the fiftieth power or made an incorrect generalization from their work with the preceding problem. For example, Clinician D presented the problem of 3^{50} divided by 4 and was interrupted by the observer who emphasized: "This is another problem. Think about it. How would you go about solving it?" C/D added: "You don't have to do it; just what would you have to do to solve it?" The observer, interjected: "What would you have to do if you had to go about solving that one?" The child responded: "So first I would figure out what 3 to the fiftieth power would be." C/D interrupted: "Would you? Is that what you did for two to the fiftieth?" Child: "Yah, I tried." C/D: "But what did you do to eventually solve it? How did you come up with the answer? Child: "I was doing these weird things in my head." Laughter followed, and the child responded with: "Well, it would be remainder one but." C/D interrupted: "Why do you think that?" Child: "Because 50 is an even number and even numbers have remainder one.."

CONCLUSIONS

Clinicians fared well in sections of the protocol that required guiding and presenting information but had considerable difficulty in allowing the child to construct a solution to the problem presented in section 2. Some prematurely jumped to section 4; others offered the simpler problems directly to the student, precluding the possibility of a spontaneous response from the child. In sections 3 and 4, some clinicians did not give the child sufficient time to think of a simpler problem but instead offered it. Hints, given too soon, did not produce learning as measured by the child's success in generalizing the heuristic to a problem of similar structure.

Despite some errors and occassional misuse of the script, teacher/clinicians were beginning to recognize that learning had indeed not occurred when the child was unable to construct the knowledge in the course of the interview and was unable to generalize to another similar problem. Clinicians' reflections and discussions on their own behavior with respect to the child appeared to lead to a deeper recognition of the constructive process by which children learn. Preliminary examination of these data suggests that changes in clinicians were beginning to occur. What requires further study is how teachers themselves conceptualize learning and how they become aware of their own and their students' learning processes.

IMPLICATIONS

The teacher/clinician, able to focus on individual learning, can become a better observer of children's learning. For teachers to deal adequately with the complexity of the classroom and learn effectively to allow children to construct their mathematical ideas and procedures, it is suggested that participation in a variety of integrated experiences as learners, clinician/observers, and philosophers be studied. As learners, they could experience learning environments parallel to those recommended for the construction of knowledge by children. As clinician/observers, they could participate in the construction of learning environments for individual and small groups of children. As philosophers, they could reflect continously on what it is they are trying to accomplish, why, how, and for whom. Then, perhaps, the transition from "direct instruction-lecture teachers" to "constructors of effective learning environments for children" might be possible.

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CHILDREN'S HEURISTIC PROCESSES IN MATHEMATICAL PROBLEM SOLVING DURING SMALL GROUP SESSIONS

BY

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ABSTRACT

The paper describes individual and group problem solving behavior of seventeen sixth grade children engaged in small group problem solving sessions to determine (1) whether and in what ways the heuristic "think of a simpler problem" is employed spontaneously, after prompting, and with guided suggestions and (2) whether the individual learner profits from the small group activity. Analysis of student responses showed that individual learning is pursued in a small group organization and that the cooperation of small groups supported the directions of individual learners.

BACKGROUND

The use of structured clinical interviews for studying children's problem solving has begun to provide useful information regarding competent problem-solving performance. Work by Goldin and Germain (1983) and Goldin (in press) regarding children's use of the heuristic process "think of a simpler problem" has provided the basis for considering the problem-solving behavior of children working in small groups and their use of the heuristic. Silver (1985) and Noddings (1985) have suggested the study of the processes of cooperative small groups for such activities as planning, monitoring, evaluating and the constructing of representations of mathematical ideas.

An early study by Alston and Maher (1984) in which small groups of mathematically able middle-school children were provided an opportunity to construct the properties of an Abelian group using concrete, symbolic, and abstract embodiments in a non-numerical context and relate the activity to work with numbers showed promise for creating effective classroom environments for children's learning.

PURPOSE

The study examines the effectiveness of organizing instruction in such a way as to allow children working in groups the opportunity to construct solutions to a problem employing the heuristic "think of a simpler problem" without direct teacher interference but with a structured group problem protocol.

DESIGN

The Problem Task

A single-person script based on the problem "What is the remainder when 2 to the 50th power is divided by 3?" (Goldin, in press) was modified for small group use in order to observe whether and in what ways the heuristic "think of a simpler problem (TSP)" might be employed. The problem task, divided into five parts, each to be administered after the former is completed, was designed to provide the children with opportunities to solve the problem (a) spontaneously, (b) with the heuristic suggested and (c) with specific examples using the heuristic given. The prerequisite skills for the TSP task, presented in Part 1, are raising a number to a power, finding the remainder when dividing a whole number by a one digit divisor and finding a remainder when dividing a number expressed in exponential form by a one digit divisor. In Part 2 presentation of the problem, "What is the remainder when 2 to the 50th power is divided by 3?", is presented without other comment. Provision for noting all the ideas emerging from the group as they seek a solution is made. In Part 3 the use of the heuristic TSP is suggested. The children are asked to generate problems that are simpler than the original problem and to solve each simpler problem, considering whether the solutions might be helpful in solving the original problem. In Part 4 the heuristic "TSP" is specifically presented by asking the children to solve a series of simpler problems structurally equivalent to the original, to organize these solutions into a chart, to look for the pattern of remainders and to give the solution to the original problem. An assessment of the depth of understanding is made by asking for the remainder when 2 to the 44th power is divided by 3 and when 2 to the 75th power is divided by 3. Finally the children are asked how they would approach the problem of finding the remainder when 3 to the 50th power is divided by 4. In Part 5 the children are asked to review what they have done.

Subjects

Seventeen eleven and twelve year old sixth grade children, five girls and twelve boys, from an independent school participated in the study. Six groups were formed, five with three children and another with two.

PROCEDURES

The 90 minute sessions, held in the school library, were audio taped with three also video taped. An observer for each group administered the problem, took notes and monitored the audio equipment, having been instructed to answer only procedural questions. Each group of children was instructed to select a recorder responsible for keeping a record of the responses to each question based on agreement of the group. Each child was given a protocol, blank sheets of paper and pencils.

Approximately 45 minutes to one hour was needed to complete the task. Analysis is based on the children's completed protocols, the work they did on other paper, observers' written notes and the audio and video tapes.

RESULTS

The data were organized according to the problem outline to describe (a) the six groups' problem solving behavior and (b) problem solving activity of the seventeen individual students.

Part 1

Results of Part 1 indicated that all of the children had the mathematical skills necessary to solve the problem. Mistakes, in one case conceptual giving 3 to the 2nd power as 6 and in a second computational responding with 3 as the remainder when 17 is divided by 5, were recognized by others in the group and corrected apparently with understanding by the individual.

Part 2

For Part 2, four of the groups attempted to compute 2 to the 50th power by multiplying the factors of 2 even though one or more individuals in each group suggested that there should be a simpler way. None of these groups reached a correct solution. In Group 5, J worked individually but in dialogue with the others to generate several attempts at simpler means of solution. Each solution was discarded, however, when the group together saw that the structure was inappropriate. The activities of individuals within the group indicated that two of the groups solved the problem in Part 2 by using a simpler problem. In Group 2, C reasoned that 2 to the 50th power is proportional to both 2 to the 10th power and 2 to the 2nd power and so must have the same remainder when divided by 3, requiring the remainder to be 1, a correct answer that was inconclusive as to understanding. In Group 5, J, leading G and L, reasoned that since 2 to the 4th power is the same as 4 to the 2nd power, 2 to the 50th power must be equal to 10 to the 10th power, which the children computed and divided by 3 for a remainder of 1. The children expressed pleasure at having constructed this "simpler problem" and indicated a preference to this method over the pattern later suggested.

Part 3

All of the groups generated simpler problems in Part 3 although only three of the groups came up with problems of similar structure to the original and Group 1 was alone in generating a series and recognizing the pattern of even and odd exponents corresponding to 1 and 2 as remainders without guidance.

Part 4

Each of the other groups recognized the pattern in Part 4 and all six predicted that 1 would be the remainder for 2 to the 44th power and 2 for 2 to the 75th power when each was divided by 3. Three of the groups responded to the question "What is the remainder when 3 to the 50th power is divided by 4?" by immediately generating simpler problems of the same structure with both even and odd exponents and finding the pattern of 1 and 3 as remainders. Group 2 and Group 5 each returned to the strategy used to solve the original problem. Group 3, the only group with an incorrect answer to the original problem even after successfully generating the pattern, seemed confused and predicted that the pattern of remainders for 3 to the 50th power would be 2,2,1,1,1,1,.....

IMPLICATIONS

During mathematics class on the next day, the children discussed the problem, compared strategies and results, and were guided by the teacher to consider proposed solutions so that misconceptions might be addressed. These results suggest that the group context can be one in which children might construct understanding of mathematical ideas. The heuristic "TSP" was proposed and considered by the children and discarded when they were not ready to use it. For some, the arithmetical representation of the problem was applied. For others, "TSP" was used spontaneously or in response to guiding questions. The group organization permitted each child freedom to pursue various paths to solution but also guidance by peers to examine those paths and to consider alternatives. Research and analysis should follow in which other problem tasks are developed and tested, presenting problems that call for various heuristic strategies to similar groups of children. Methods need to be developed to compare group problem solving activity with that of children working individually and in whole class activities in order to make intelligent inferences for classroom instruction.

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A CHRONOMETRIC ANALYSIS OF ADDITION AND SUBTRACTION PROCESSES USING POSITIVE AND NEGATIVE NUMBERS *

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Reaction times (RT) of college students majoring in technical fields were measured in simple addition and subtraction tasks. Subjects determined the truth value of mathematical statements of the type $(-2)+(+6)=(-4)$*. Results reveal RT patterns similar to those obtained in experiments outside the domain of mathematics. Findings also reveal that Anglos were significantly faster than Hispanics. A clustering of RTs according to statement types suggests efficient processing strategies, such as combining the two -'s into an overall + in statements such as* $(+6)-(-2)=+8$*.*

In recent years, a considerable number of research studies have focused on understanding mental processes involved in the addition and subtraction of simple numbers. These studies fall into three major types. One type of study uses the interview approach. The subject "thinks aloud" while he/she is engaged in some addition or subtraction task, and the resulting record of the interview, called the "protocol," is analyzed to identify the processes that the subject used. Studies of this type have been successful in identifying counting procedures of varying degrees of sophistication as well as stages by which children make the transition from less, to more sophisticated procedures (Carpenter & Moser, 1984; Fuson, 1984).

A second type of study attempts to test models by implementing them on computers. The success of these models is measured by the extent to which they are capable of predicting the types and frequency of use of both successful and erroneous strategies. Some of these models are not only quite elaborate, but are capable of "solving" a wide range of addition and subtraction problems (Briars & Larkin, 1984; Riley, Greeno & Heller, 1983). However, some researchers (Carpenter & Moser, 1984) argue that these precise models do not capture the variability of children's performance.

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* Work supported by National Institute of Education Grant #G-83-0072. The contents herein do not necessarily reflect the position or policy of NIE.

In the third type of study, the insights come from analyzing the "reaction time" (RT) it takes subjects to perform an addition or subtraction task. Generally the tasks in RT studies with young children consist of pressing a key to designate the answer to a problem (e.g. pressing the "8" key in an array of labeled keys when given the problem "3+5"). With older children and adults, the tasks generally consist of verifying an equation (e.g. pressing a "true" or "false" key in response to a stimulus such as "3+5=8"). The advantage of the RT approach is that it can potentially provide insights into mental processes involved in tasks that are generally performed correctly by adults (limiting the usefulness of analyzing error patterns) and tasks that are highly automated (making it very difficult to elicit subjective, introspective reports). The difficulty in a RT analysis comes in deciding which processes should take a constant amount of time regardless of variations among the task (e.g. representing numbers, such as "3", in memory), and which process(es) varies in time as a function of variations among the tasks. The time-varying process is designated by a "structure variable" which is then fit to the RT data. In a seminal RT study, Groen and Parkman (1972) determined that the best structure variable in predicting RT in problems such as m+n=s was min(m,n) which implies the use of a "count on from larger" strategy. Reaction time studies have been quite successful at identifying important structure variables in both addition and subtraction (Ashcraft, 1982; Woods, Groen & Resnick, 1975).

One area which has not been investigated in RT studies is the effect of manipulating mathematical operations upon performance. For example, if we allow both negative and positive integers within the context of addition and subtraction, we can usually construct several three-digit mathematical statements which are equivalent to each other. The following cases illustrate three different manipulations on the digits 3 and 4 that maintain the overall result equal to negative one: "(+3)+(-4)=(-1)", "(+3)-(+4)=(-1)", and "(-4)+(+3)=(-1)". It may be, however, that among these equivalent equations, certain ones may be considerably easier to process than others. One question that can be answered by comparing the RTs required to verify equations in the example above is whether adding a positive and a negative number is easier or harder to carry out than subtracting the corresponding two positive numbers.

Another important question that we can investigate with such manipulations is whether there are differences between the RT patterns of Anglos and Hispanics in verifying mathematical statements containing two negations, such as "$(+3)-(-4)=(+7)$". The significance of this question comes from the fact that in the Spanish language, certain double-negative constructions retain an overall negative meaning, rather than reverting to a positive meaning as is the case in grammatically correct English. We have investigated this question within verbal comprehension tasks using different number and types of negations (Mestre, 1984; in press) and found that the overall performance of Anglos was better and significantly faster than that of Hispanics. However, because of generally poor performance by both groups, there was no clear evidence that Hispanics were particularly worse than Anglos on the double-negative tasks. We thus think it would be interesting to look for Hispanic-Anglo differences in double-negative performance within a mathematical context.

Finally, are we not aware of any manipulations of the affirmation-denial dimension within a RT math study; this would mean investigating the differences in RT patterns between "$=$" statements (affirmations) and "$\neq$" statements (denials). There have been affirmation-denial studies outside the domain of mathematics. In a sentence verification task, Carpenter and Just (1975) measured RTs of subjects as they determined the truth value in a situation where a simple sentence was followed by a picture. They found that RT increased in the following order: True affirmatives (e.g. "The dots are red" followed by a picture of red dots), false affirmatives, false denials, and true denials (e.g. "The dots aren't red" followed by a picture of black dots). It would therefore prove interesting to see if this same pattern emerges within a mathematical context.

In this paper we report a preliminary analysis of a RT study investigating the effect of manipulating equality as well as addition/subtraction operations among Anglo and Hispanic college students.

Procedure

Subjects: A total of 58 subjects participated in the experiment. All subjects were majoring in technical fields such as engineering, math and chemistry. Of the 58 subjects, 27 were Anglo monolingual speakers of English and 31 were bilingual Hispanics. Subjects were paid for their participation in the study.

Tasks: Reaction times were measured as subjects determined the truth value of mathematical statements of the following forms:

$$(\pm 6)\pm(\pm 2) = \pm\genfrac{}{}{0pt}{}{8}{4}\ ,\quad (\pm 6)\pm(\pm 2) \neq \pm\genfrac{}{}{0pt}{}{8}{4}\ ,\quad (\pm 2)\pm(\pm 6) = \pm\genfrac{}{}{0pt}{}{8}{4}\ ,\quad (\pm 2)\pm(\pm 6) \neq \pm\genfrac{}{}{0pt}{}{8}{4}$$

A total of 128 such statements can be constructed, 64 with "=" and 64 with "≠". Of the 64 statements with "=", 16 are true and 48 are false, while of the 64 statements with "≠", 16 are false and 48 are true. To balance the number of true and false cases from the "=" statements, an additional 32 true cases were included in the pool of statements; similarly, 32 additional false cases were included in the pool of "≠" statements. Each subject therefore received a total of 192 statements.

Method: Subjects answered "true" or "false" by pressing one of two keys with the index fingers of both hands; the "true" key was assigned to the dominant hand. Statements were randomly selected and presented on the screen of an Apple II-E microcomputer and a Mountain Clock recorded the RTs with millisecond accuracy. Only correct responses were kept for the analysis. Errors were recycled into the pool of statements so that every subject eventually answered all 192 statements correctly. Subjects were instructed to move as quickly as possible without sacrificing accuracy. Statements were presented in blocks of 24, with feedback about speed and accuracy given after each block. Two practice blocks preceded the actual experiment.

Results

The error rates for Anglos and Hispanics were 6.7% and 8.2%, respectively. To a first approximation, the patterns of RTs were quite similar for Anglos and Hispanics, although Hispanics in our sample averaged about 600 msec longer per response. Averaging over trials, we find the same order of increasing RT for both Hispanics and Anglos as in the sentence verification study of Carpenter and Just, although there were significant differences in overall speed between Anglos and Hispanics (see table below). A Group (Hispanic vs. Anglo) x Equality (= vs. ≠) x Truth Value (T vs. F) x Statement Type analysis of variance was conducted on the RT data. By Statement Type we mean the 8 possible orderings of the three signs on the left hand side of the statement, namely (+++), (++-), (+-+), (-++), (+--), (-+-), (--+), and (---). All main effects were significant

		True	False
=	Anglo	2.29	2.65
	Hispanic	2.77	3.24
≠	Anglo	2.97	2.84
	Hispanic	3.61	3.46

($p < .001$). This means that 1) Anglos performed significantly faster than Hispanics, 2) The ≠ statements took significantly longer to process than the = statements, 3) False statements took significantly longer to process than True statements, and 4) There were significant differences in processing times among the 8 Statement Types. There were two significant interactions (both $p < .001$): 1) Equality x Truth Value (see table above) indicating that the difference between true and false RTs was much different for = and ≠ statements, and 2) Equality x Truth Value x Statement Type indicating that the pattern of RTs for the 8 Statement Types varied across the four possible combinations of Equality x Truth Value. There was no evidence that the pattern of RTs for statements involving one and two -'s were different for Anglos and Hispanics. However, the difference in RT between Hispanics and Anglos was significantly smaller ($p < .05$) for the (+++) statements than for the average of the 7 Statement Types containing at least one -.

Statement Type RTs broke down into four clusters that were nearly the same across all four Equality x Truth Value cases for both Anglos and Hispanics. The table below shows this clustering where we have averaged over Equality and Truth Value. With only one exception each for Hispanics and Anglos, Statement Types (+++) were processed significantly faster ($p < .05$) than the cluster of Statement Types (++-), (+--) and (-+-), which in turn were processed significantly faster than the cluster (+-+), (-++) and (--+), which in turn were processed significantly faster than the Statement Type (---); the two exceptions were that (-+-) was not significantly different from (+-+) for Anglos, and (--+) was not significantly different than (---) for Hispanics.

	(+++)	(+--)	(-+-)	(++-)	(+-+)	(-++)	(--+)	(---)
Hispanics	2.32	3.24	3.26	3.29	3.49	3.47	3.47	3.64
Anglos	1.93	2.63	2.64	2.64	2.77	2.84	2.92	3.11
Difference	.39	.61	.62	.65	.72	.63	.55	.53

Finally, including the order in which the digits "2" and "6" appeared on the left hand side of the statements in the ANOVA resulted in a significant main effect ($p < .01$) predominantly due to the faster RTs in the subtraction

Statement Types (+-+) and (++-) when the "6" preceded the "2".

Discussion

The presence of various statistically significant main effects as well as significant interactions in our analysis indicates that measures of RT are a good means for unravelling the mental operations and processes used by adults in performing simple arithmetic manipulations. Although we have not as yet explored the predictive ability of various models for our observed RT patterns, it might prove interesting to speculate on the mental processes which might account for some of our findings. The processing time for the tasks of this study can be subdivided into four stages: 1) Encoding, 2) Computing, 3) Verifying, and 4) Responding. Since Encoding and Responding are likely to take the same amount of time for each trial, the variation among RTs comes from the Computation and Verifying stages. By averaging over all Statement Types, the processing time can be collapsed into the Verifying stage for the four Truth Value x Equality cases (i.e. T=, T≠, F=, and F≠). The fact that our observed RT pattern for these four cases is the same as that in the Carpenter and Just (1975) study, despite a lack of parallelism between our Equality dimension and their affirmation-denial dimension, suggests that subjects used a similar verification process in both studies.

A successful model for the Computing stage would have to explain the clustering among the Statement Types observed in the table above. For example, the data indicate that latency is not simply determined by the number of -'s in the statement, but rather by the efficiency with which the various operations can be combined to yield an answer. As evidence, consider the relatively fast RTs on (+--) which suggest that the double negative is encoded as an overall positive. The data also suggest that processing is relatively fast for "adding like numbers", as in the cases (+++) and (-+-). Another general trend is that cases in which the - is at the end of the statement are processed faster than cases in which the - appears in the middle or at the beginning of the statement (i.e. (++-) is processed faster than (+-+) and (-++)). Finally, the faster latencies when the "6" preceded the "2" in the (++-) and (+-+) Statement Types indicates that subtracting a "2" from a "6" is more "ecologically natural" than subtracting a "6" from a "2".

The fact that the RT pattern for Hispanics did not differ appreciably from that of Anglos on Statement Types containing two negatives indicates that the lack of parallelism between Spanish and English in the meaning of double-negations does not interfere with arithmetic performance. It was surprising that on the simple tasks administered Hispanics took approximately 600 msec longer than Anglos, especially since all subjects were technical majors; the cause of this large difference in speed remains a mystery. We will only remark that for Hispanics, SAT scores were negatively correlated with RT ($p < .05$) implying higher SAT scores for those Hispanics with faster RTs; this suggests that SAT scores may be a poor predictor of academic performance because Hispanics proceed at a slower pace and therefore complete fewer items. Further support of this conjecture comes from the fact that, despite Anglos scoring approximately 200 points above Hispanics on both Verbal and Math SATs, their GPAs did not reflect this large difference (GPA_H=2.46, GPA_A=2.94 out of a possible 4.0).

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PREDICTING MATHEMATICS ACHIEVEMENT FOR FEMALES AND MALES FROM CAUSAL ATTRIBUTIONS

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The study reported here investigated the relationship between causal attributions as measured by the Mathematics Attribution Scale in grade 8 and achievement in grade 11. Correlations and linear regressions were done separately for females and males in order to see if the relationships differed by sex. Success Ability and Failure Ability were correlated with achievement for both females and males. Grade 8 attributions were more important for females in predicting grade 11 achievement. A score for mastery orientation / learned helplessness based upon attributions was correlated with achievement for males.

Causal attributions, or the reasons students give for their successes or failures, have been the subject of study in recent years as they relate to achievement in mathematics (Wolleat, Pedro, Becker, & Fennema, 1980; Eccles, Meece, Adler, & Kaczala, 1982). Much of the appeal of these variables lies in the intuitive connections that can be made between various attributions and subsequent achievement related behaviors. Causal attributions are also of interest in attempts to explain sex-related differences in mathematics achievement. There is some evidence that females and males attribute causation differently and that these differences are related to achievement differences (Wolleat et al, 1980; Eccles, 1983).

Two of the problems associated with causal attribution research are those of measurement and interpretation. Typically, attribution instruments are designed to reflect the Weiner (1974) model with subscales written to measure student perceptions of the importance of ability, effort, luck and task difficulty for their success or failure. It has proven difficult to sort out relationships when 8 subscales are involved. Some research has attempted to identify which of the subscales might be more important relative to achievement. Combining subscale scores is another way that has been used to simplify using causal attributions. Fennema and Peterson (1984) derived a formula based on constructs of learned helplessness and mastery orientation. The formula was defined as follows:

$$MO/LH = (SA + SE + FE) - (FA + ST + SO)$$

The theory based formula suggests that mastery orientated students perceive that their successes are a result of their ability (SA) and effort (SE) and that their failures are due to a lack of sufficient effort (FE). In contrast, learned helpless students perceive their successes as being the result of either task ease (ST) or help from others (SO) and their failures as resulting from their lack of ability (FA). Kloosterman (1984) used this formula in a study of Algebra I students. (See paper by Kloosterman in this proceedings.) Another limitation of studies involving causal attributions related to sex differences in mathematics achievement is that the data are usually collected at a single point in time. As result, the ability to predict future achievement based upon attributions is limited.

The study reported here attempted to address some of the issues just discussed. Specifically, the objective was to see if causal attributions measured in grade 8 could predict mathematics achievement four years later. Females and males were considered separately in order to look for sex differences in the predictions. The MO/LH score formula discussed above was also used to test if its relationship to achievement was stronger than those of the individual subscales.

DATA SOURCES AND PROCEDURES

Causal attributions and mathematics achievement data were collected on 151 students in grade 8 in a Midwestern city. The sample included 84 females and 67 males. The instruments used to measure achievement were the Basic

Concepts (Level I) and Computation tests of the STEP Basic Assessment Tests. Causal attributions were measured by the Mathematics Attribution Scale (MAS) (Fennema, Wolleat, & Pedro, 1979). The MAS contains eight subscales: Success Ability (SA), Success Effort (SE), Success Others (SO), Success Task (ST), Failure Ability (FA), Failure Effort (FE), Failure Others (FO), and Failure Task (FT). The "Others" category is an expansion of Weiner's "Luck" category and it includes other unstable, external factors like help from others. Each subscale score can range from 4 to 20. At the end of grade 11, achievement was again measured using the STEP Computation and Concepts (Level J) tests. The cognitive level of the items was used to derive three scores from the STEP Concepts test: a low level score (0-24); a high level score (0-24) and ; a total score (0-48).

RESULTS

Table 1 contains correlations and descriptive statistics for females and males for grade 8 achievement and attribution scores. In grade 8, the mean score for the males was significantly higher than that of the females on both STEP Computation and STEP Concepts. In terms of attributions, the only significant differences by sex were on the subscales Success Task (ST) and Failure Task (FT). Mean scores on these subscales indicated that females were more likely than were males to focus on the task as a reason for both success and failure. Considering the grade 11 achievement measures, the mean scores for the males were significantly higher than those for the females on STEP Concepts High and STEP Concepts Total.

Success Ability (SA) and Failure Ability (FA) were the two attributions in grade 8 that were most consistently correlated with achievement in grade 11. for both females and males. Success Effort (SE) and Failure Effort (FE) were also significantly negatively correlated with STEP Concepts Total and STEP Concepts High, but only for the females. For the males, achievement on STEP Concepts Total and STEP Concepts High were significantly correlated with the attribution of Failure to Others (FO). Somewhat surprisingly, MO/LH was

Table 1

Descriptive Statistics and Correlations for Grade 8 Achievement and Causal Attributions and grade 11 Achievement.

		COMP12	CONC12	CONCL12	CONCH12	MEANS	
COMP8	F	.49 ***	.49 ***	.47 ***	.46 ***	<43.38>^{a}	(10.24)
	M	.67 ***	.62 ***	.53 ***	.59 ***	<46.58>	(6.17)
CONC8	F	.58 ***	.51 ***	.56 ***	.43 ***	<37.80>	(9.28)
	M	.70 ***	.61 ***	.47 ***	.61 ***	<42.10>	(4.72)
SA8	F	.24 *	.27 **	.20 *	.28 **	13.69	(3.12)
	M	.27 *	.41 ***	.27 *	.44 ***	14.25	(3.15)
SE8	F	-.17	-.21 *	-.13	<-.24 *>	12.64	(3.92)
	M	-.02	.11	.11	< .11 >	12.73	(3.70)
SO8	F	.12	.04	.03	.04	14.33	(3.04)
	M	-.09	-.04	-.10	.02	13.64	(2.71)
ST8	F	-.13	-.06	-.05	-.05	<14.70>	(2.47)
	M	-.18	-.20	-.20	-.16	<13.88>	(2.63)
FA8	F	-.23 *	-.28 **	-.18	-.30 **	10.67	(2.85)
	M	-.25 *	-.37 **	-.32 **	-.34 **	10.64	(3.13)
FE8	F	-.29 **	-.27 **	-.17	-.30 **	11.99	(3.51)
	M	-.02	-.03	.00	-.04	11.76	(3.05)
FO8	F	-.07	-.11	-.07	-.12	12.26	(2.99)
	M	-.05	-.25 *	-.15	-.28 *	12.00	(2.97)
FT8	F	-.05	-.12	-.06	-.15	<14.12>	(2.67)
	M	-.06	-.15	-.09	-.17	<12.84>	(2.76)
MO/LH8	F	<.00 >	<.02 >	<.03 >	<.01 >	-.31	(2.43)
	M	<.31 ** >	<.45 *** >	<.41 *** >	<.41 *** >	.39	(2.46)
MEANS (s.d.)	F	49.52 (4.90)	<38.94> (6.96)	21.03 (2.88)	<17.90> (4.56)		
	M	50.03 (5.03)	<41.74> (5.24)	21.85 (2.31)	<19.87> (3.46)		

* $p<.05$, ** $p<.01$, *** $p<.001$;^{a} brackets indicate significant diff. $p<.05$.

significantly correlated with grade 11 achievement only for the males. None of the correlations for the females were significantly different from zero.

To investigate the predictive ability of the attribution variables, each of the four grade 11 achievement variables was regressed on the 8 attribution scores from grade 8. The two grade 8 achievement scores were also used as predictor variables in order to see the contribution of the attribution scores independent of prior achievement. For the males, attribution scores contributed to the accounted for variance only in the equation for STEP Concepts High. Failure Others (FO) added .08 to the accounted for variance and Success Ability added .03. For the Females, attributions entered each of the four equations for grade 11 achievement. Failure Ability (FA) added .09, .04, and .03 to the variance accounted for in the equations for STEP Concepts Total , Concepts Low and Concepts High respectively. Failure Effort (FE) added .10 and .13 to the variance accounted for in the equations for Computation and STEP Concepts High. In separate analyses, MO/LH added .07 to the variance accounted for in a regression equation for Concepts High when it was regressed on grade 8 achievement and MO/LH for females. MO/LH did not enter any of the regression equations for males.

CONCLUSIONS AND DISCUSSION

Based upon the results presented above, several tentative conclusions can be drawn. First of all it was clear that the attribution subscales were not all related to achievement equally well. Success Ability and Failure Ability were the two most strongly correlated with achievement for both females and males. The MO/LH score was significantly correlated with achievement only for the males, but it still might be important in predicting achievement for females as evidenced by its contribution to a regression equation predicting Concepts High for females. The results of the regression analyses suggest that for the males causal attributions might not be important as predictors of future achievement independent of prior achievement. However for the females, attributions do have predictive ability independent of prior achievement and for this reason further research is indicated. The differing results for both females and males based upon cognitive level suggest that this classification is important and that cognitive level should be a factor in future research. In summary, although it is far from clear what role causal attributions play in determining achievement for females and males, it is clear that they do play a role and that perhaps this role is more important for females.

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PRESERVICE ELEMENTARY TEACHERS
THEIR ATTITUDES TOWARDS MATHEMATICS AND THE ANXIETIES THEY SUFFER

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An instrument designed to measure math anxiety was administered to a group of preservice elementary teachers. The results indicated that their math anxiety was no greater than the "average" adult's. However, the results of two less formal assessments indicated that some popular beliefs may need closer scrutinizing.

INTRODUCTION

Since September, 1976, when MS Magazine published an article by Shelia Tobias entitled "Math Anxiety," the term "math anxiety" has become a popular expression in social and academic circles. Fear of mathematics, avoidance of mathematics, even poor attitude toward mathematics are often associated with the popular term. Many people point an accusing finger at our nation's elementary school teachers as the source of mathematics anxiety developed in children at an early age. The purpose of this study was twofold: (1) to ascertain to what extent a select group of preservice elementary teachers suffer with math anxiety; and, (2) to assess their feelings and attitudes towards mathematics.

THE INVESTIGATION

During the 1984-85 school year, seventy students from a mid-western university and a private college participated in the study. The students were enrolled in a math methods class for elementary teachers. All seventy students were administered the Mathematics Anxiety Rating Scale (MARS) constructed by Frank Richardson and Richard Suinn (available through the Rocky Mountain Behavioral

Sciences Institute, Fort Collins, Colorado); and, a 25-item True/False attitude assessment instrument designed by Jim Daniels at the University of Texas. Fifty-eight of these seventy students also completed a 10-item sentence completion exercise designed to assess a person's beliefs about mathematics.

MATHEMATICS ANXIETY

Math anxiety involves "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations." (Richardson and Suinn, 1972) The norm mean score for the adult form of the MARS is 215 with a standard deviation of 65. The mean score for the 70 participants in this study was 221 with a standard deviation of 61. Using the Cochran and Cox method to test the significant difference between two means resulted in no significant difference between the mean MARS score of the participants in this study and the norm mean score. (Ferguson, 1976) The conclusion drawn at this point was that this group of preservice elementary teachers did not have a significant greater degree of math anxiety than the "average" adult. On the surface, this is a pleasing conclusion. However, the participants' responses to some of the items on the True/False instrument and the sentence completion exercise warrant further consideration.

A review of the literature revealed that people with math anxiety range in age from nine to sixty-five. In one study, nine to eleven year-old children, who were underachieving in mathematics, demonstrated that anxiety was the most significant contributor. (Sepie and Keeling, 1979) To prevent maximum damage to a student's self-concept, math anxiety must be conquered in the early years of intellectual development. A teacher's attitude is a potent force in the classroom. One conclusion drawn from a survey of 124 dissertations written from 1969-75 was that teachers' attitudes and their enthusiasm toward a subject have greater impact on students' attitudes than instructional variables do. (Burton, 1979)

The general attitude exhibited by the participants in this study seemed to be negative. The two statements of the sentence completion exercise which received the greatest percentage of negative responses were: "When it comes to math, I . . . " and "Doing math makes me feel . . . " Sixty percent of the responses to the first statement were of a negative nature and seventy percent of the responses to the second statement were negative.

Doyle and Graesser (1978) conducted a study using math anxious and math comfortable college students in which they were trying to degermine if the math anxious students exhibited characteristics distinguishable from the math comfortable students. One trait deemed characteristic of highly math anxious students is that they express the belief that the problem they are trying to solve has a simple solution, but that they are too dumb to see it.

The responses to two statements on the True/False assessment instrument combined with one item from the sentence completion exercise support Doyle's and Graesser's hypothesis. Statement: Good math solutions are usually complicated. Thirteen answered true, 57 answered false. Statement: Some math problems are just plain easy. Sixty-three answered true, 7 answered false. Complete: Doing math makes me feel . . . Seventy percent of the responses were negative. If a person generally feels that solutions are not complicated and problems are easy, then of course they are going to feel dumb if they cannot "get it."

A review of the literature revealed that the development of attitudes toward math is a summatory phenomenon with each conditioning experience building upon the one that precedes it. The initial attitudes seem to be affected by all the teachers of mathematics with whom the student is associated. Pupils who have done poorly or failed math have deflated egos and therefore, tend to develop attitudes of dislike and hostility toward math. Indicative of findings reported throughout the literature, a poor attitude seems to breed math anxiety.

WHAT CAN BE DONE TO PREVENT AND/OR CURE MATH ANXIETY?

There are two positive facets about math anxiety; it is curable at any stage; and, its hold is never irreversible. Research conducted in the math anxiety program at the University of Minnesota supports the hypothesis of past experiences being a contributing factor to math anxiety. (Mathison, 1977) These past experiences are generally associated with the effect of teacher influence. Math teachers have a reputation of being hard. One item on the sentence completion exercise supports this attitude. Complete: Math teachers are . . . Responses: "insensitive," "strict and not very understanding," "intimidating to me," "to smart to teach it," "hard for me to communicate with," etc. (Thirty-three percent of the responses were of this negative nature.)

Faculty in teacher education colleges can help to dispel this image of math teachers. They can also do much to resolve the dilema for preservice elementary teachers who are adversely effected by mathematics anxiety. They can provide a mechanism to diagnose math anxiety and then provide support groups, math classes, and tutorial sessions to help dispel whatever is causing the anxiety. Within college classrooms, teachers can build an atmosphere in which students are not afraid to ask "dumb questions" and then encourage them to do the same when they become teachers. Math content should be taught using methods with which students can identify. Preservice elementary teachers should be given a strong foundation in how to use manipulatives and concrete examples in their future classrooms. They should be encouraged to talk about personal math difficulties and allowed to work together on challenging problems.

SUMMARY

Math anxiety is a threat to our society's intellectual advancement. Teachers are a very important educational influence on students' learning mathematics. Therefore, educators should start early, in the formative years, to conquer math anxiety. College and university personnel should become more attuned to the anxieties and attitudes of preservice elementary teachers. They must address their students' needs in being prepared to prevent and/or fight the anxieties of their future students and to instill an improved attitude toward mathematics in general.

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COMPUTER PROGRAMMING AND MATHEMATICAL THINKING

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The effects of computer programming instruction on mathematical thinking skills and development of fundamental concepts were investigated. After six weeks of programming instruction, two randomly selected sixth grade treatment groups were compared to a control group on posttests measuring programming ability, generalization, and understanding of variables. For each dependent variable, it was determined that the average of the mean scores of the groups receiving programming instruction was significantly greater than the mean score of the control group.

If the present trend continues, there will be two million computers in the U.S. public schools by 1988. The proliferation of computers in schools is partially due to the implicit belief that computers are powerful educational tools. However, although computer programming advocates have argued its efficacy in terms of both academic and attitudinal benefits, to date there is a disturbing lack of empirical evidence substantiating these arguments.

The potential positive effects computer programming could have on mathematics learning are numerous and varied, ranging from the enrichment of mathematical concept learning to the enhancement of deductive and inductive reasoning. Proponents of computers often cite apparent relationships between certain mathematical cognitive processes, such as generalization, and the processes involved in computer programming (e.g. Hatfield, 1984; Papert, 1980). Hatfield, for instance, described the process of programming as "successive approximation" since a programmer often solves a problem for a restricted set of data first and then extends and modifies the program for a larger universe. Hatfield asserts that this refinement and extension of already successful programs will foster thinking strategies such as generalizing and conjecturing.

Unfortunately, most claims of a relationship between mathematical thinking and cognitive processes used during computer programming have been based on rational argument, individual observation, and the experience of "expert witnesses" -- such as practitioners and educators — and not supported by systematic empirical research. The few mathematics education studies of the cognitive consequences of learning to program had only marginally positive results (Milojkovic, 1983; Foster, 1972). Several psychological studies comparing expert and novice programmers seem to indicate a relationship between programming ability and mathematical generalization (e.g. Pea and Kurland, 1984; Jefferies, 1982).

In light of the limited but positive empirical results, the intent of my research was to investigate the effects of computer programming instruction on one specific mathematical thinking process; namely, generalization. Since computer programming and expression of mathematical generalizations rely extensively on the use of variables, a second facet of my study was an investigation of elementary students' understanding of variables and the relationship of this understanding to the students' mathematical generalization process and computer programming ability.

While generalization is only one of the mental processes that correlates positively with mathematical aptitude, many mathematicians feel it is an extremely important attribute of mathematical maturity (e.g. Mason, 1982; Krutetskii, 1976). Mason refers to generalization as "the life-blood of mathematics (p. 9)" while Krutetskii states that "abstractions and generalizations constitute the essence of mathematics and mathematical thinking (p. 86)."

PROCEDURE

As previously mentioned, my investigation focussed on the effects of computer programming instruction on sixth grade students' mathematical generalization ability and understanding of variables. Sixth grade classes of two elementary schools were the treatment groups. The classes of one school acted as a control (Group C) while the students of the other were randomly assigned to experimental group W or experimental group E. For 6 weeks, both groups W and E studied BASIC computer programming during 60-90 minute sessions 2 or 3 times a week. Overall, these students participated in approximately 20 hours of programming instruction. Pre- and posttests assessing programming ability (PROG), mathematical generalization (GEN), and understanding of variables (VAR) were administered. Table 1 is a summary of the three treatment groups and includes means and standard deviations of the pre- and posttests.

Table 1

Summary of Treatment Groups

		Group		
		W	E	C
Subjects (N)		26	29	29
Sex (M/F)		12/14	14/15	15/14
Ability		54.5	52.6	52.2
GEN	Pre	13.4 (5.3)	14.4 (6.5)	11.6 (5.9)
(29)	Post	15.0 (6.5)	14.6 (6.2)	11.9 (6.5)
PROG	Pre	2.6 (4.1)	2.3 (2.1)	2.8 (2.2)
(29)	Post	9.5 (5.0)	8.8 (5.0)	3.1 (4.6)
VAR	Pre	2.2 (1.4)	2.4 (1.6)	2.9 (2.7)
(14)	Post	2.8 (2.3)	3.8 (2.2)	1.8 (1.5)

Integration of computer programming into the elementary mathematics curriculum is still in its infancy. Thus the purpose of two separate experimental groups was to study the effects of two different instructional methods. The Wholistic approach (Treatment W) began instruction at the whole program level. The focus was on mathematically relevant problems with commands introduced only as needed to solve the problem. On the other hand, the Elemental approach (Treatment E) focussed on the individual BASIC commands and proceeded stepwise until the students were capable of programming complex problems. Further explanations of these two treatments can be found in Oprea (1984).

In accordance with Dienes (1961) operational definition of generalization, I developed a paper and pencil instrument (with items similar to Krutetskii's interview questions) to measure generalization ability. This generalization instrument, with a reliability of .80 (Cronbach's α, $N=78$), consisted of four subtests, each of which involved one of the following mathematical concepts: combinations, exponents, number patterns, and geometrical shapes. The items of each subtest were ordered and weighted so that hypothetically the number of items successfully completed indicated the student's level of generalization. Instruments measuring programming ability (reliability = .82) and understanding of variables (reliability = .67) were also developed.

RESULTS

Multivariate statistical methods were used to measure the effects of the treatment on the three groups. Using Mathematical Ability (as measured by the Mathematical Application subtest of the CTBS) as a concomitant variable, multivariate analysis of covariance was calculated and found to be statistically significant (Wilk's lambda = .667, approximate $F = 4.81$, $p < .0002$). Follow-up analysis included individual ANOVAs using approximated mean squares for the three dependent variables and Dunn's multiple comparison test. For each dependent variable, it was determined that the average posttest mean of the groups receiving programming instruction was significantly greater than the posttest mean of the control group. The alpha levels were as follows: Generalization ($p < .1$), Programming ($p < .005$), and Understanding of Variables ($p < .05$). Based on these statistical results, the following conclusions can be drawn:

1. Sixth grade students can learn to program.
2. Learning computer programming enhances sixth grade students' understanding of variables.
3. There is preliminary evidence that programming instruction enhances sixth grade students' mathematical generalization.
4. The researcher was unable to substantiate the claim that different instructional methods would influence

the student's mathematical generalization, programming ability, and understanding of variables.

CONCLUSIONS

From the considerable number of students that are being taught computer programming, one might conclude that programming aptitude and skills can be developed in any student at any grade level. Yet there is little mathematics education research to support this assumption. Thus my study served to verify this contention for sixth grade students. This result also has pragmatic implications since among educators or practitioners, there is no accepted level of programming competency. Inasmuch as the programming instrument proved reliable and valid, these results can serve as a measuring stick in future research and curricular development.

My research addressed the issue of whether computer programming promotes the development of thinking skills. Inasmuch as the statistical results were marginally significant, my research can be considered preliminary support of this hypothesis. Since the research theories about learning and teaching computer programming in elementary schools are still evolving, it is probably premature to draw definitive conclusions regarding the effects of computer programming instruction on mathematical generalization. Yet, the positive — although marginal — results and relative importance of this issue justifies further investigation.

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DECIMAL CONCEPTS AND OPERATIONS: WHAT DO STUDENTS THINK?

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Ninety-six students in grades six, seven, and eight took a test of 33 free response items on various aspects of decimal concepts, computations, and problems, but emphasizing multiplication in particular. Following the written test 15 students were interviewed to identify rationales and probe students' understanding of decimals as numbers. Performance was generally acceptable on computation. Students rely on rules they have been taught, rather than reasoning, for example in an estimation task. Other conceptual tasks of translating from words to numerals and supplying a number between two decimal numbers were troublesome.

For at least 50 years mathematics educators have been concerned about the quality of students' understanding of mathematics as well as being able to cipher. The CSMS group in England (Hart, 1981) concluded that work with decimals is not as simple as recalling place names and rules for computation. Rather a whole series of relationships is involved in the integration of decimals as numbers into the system. In their investigations Hiebert and Wearne (1983) concluded that students have generally created few links between form and understanding. From a series of tasks focusing on the meaning of decimals in various contexts, they concluded that students are more influenced by form than understanding in making decisions.

The purpose of this paper is to investigate the relationship between performance on computation, especially multiplication of decimals, and the more conceptual notions of estimation of product, translation from words to symbols, and naming a number between two decimal numbers. It is hypothesized that students' performance on computational procedures will outstrip their understanding of decimals as numbers.

METHODS

The study involved 24 grade six, 26 grade seven, and 46 grade eight students at two schools in the Greater Vancouver area. In British Columbia elementary schools go through grade seven. The two classes of grade eights were in a junior secondary school.

A test of 33 free response items was constructed. The Addition, Subtraction and Multiplication computation items were presented in horizontal format. The numbers were chosen to minimize the difficulty with facts and whole number algorithms. The Number Line items asked students to name a marked point between two designated points. The Order items required ordering three given numbers. The Problems were one-step applications such as average speed, cost of gasoline, fuel consumption and enlargement. The remaining tests will be described and examined in detail in the Results section.

On the basis of the written test and teacher recommendation, students with a range of competency were selected for interviews. Originally six from each grade were selected, but five from each grade had usable transcripts of the interviews. The informal interviews, conducted by a graduate assistant, varied in length from 40 minutes to an hour. The interview format was adjusted to the subject's written test performance and was designed to ascertain computational strategies and probe understanding of decimals as numbers.

RESULTS AND CONCLUSIONS

Descriptive data for the written tests are displayed in Table 1. While particularly difficult computations were avoided, the results confirm that generally these students had a good grasp of setting up and placing the decimal point in the computed result. Decimal numbers were not a particular obstacle to solving the application problems in multiplication settings. In retrospect the quality of the Problems test would be improved by at least one divide item and at least one less obvious multiplication such as a reduction situation. Ragged decimals in the Number Line and Order subtests would have been more difficult and more revealing of understanding.

TABLE 1

MEANS OF THREE GRADES ON EIGHT WRITTEN SUBTESTS

	No. of Items	Grade 6 N=24	Grade 7 N=26	Grade 8 N=46
ADD AND SUBTRACT	5	3.70	3.58	4.54
MULTIPLY	10	7.38	7.23	9.20
PLACE DECIMAL	3	1.04	1.77	1.46
TRANSLATE & MULTIPLY	2	1.00	.73	.76
NUMBER LINE	2	1.33	1.54	1.39
ORDER	2	1.25	1.69	1.52
PROBLEMS	6	4.29	4.12	4.22
BETWEEN	3	.92	1.00	1.13

One item from each of the remaining subtests has been chosen to detail from the interview data. Item 17 was stated: Estimate the answer, and place the decimal point in the given "answer": 3.25 x 6.25 = 2119. Item 16 was similar. While

Item 18 required placing the decimal point in one of the factors, a "count the decimal places" strategy was sufficient. As a result Item 18 was easiest. Results of the interviews on Item 17 are summarized in Table 2. Of the 15 students interviewed only one was successful on the Item 17 of the written test and only four during the interview. All interviewees except one went through some form of estimating, but most were unable to make use of it. It is clear that the overriding

TABLE 2

INTERVIEW SUMMARY FOR "PLACE THE DECIMAL POINT"
ITEM NUMBER 17

GRADE SIX	GRADE SEVEN	GRADE EIGHT
Tony 14,0,0* 7x1=7: .3671 Agreed estimation may help. He couldn't explain.	Max 9,1,0 When you can't line up decimals you count: .3671	Rhoda 15,1,0 7x1: Alternated between 3.761 and 36.71. "I can't remember."
Carl 16,1,0 "5 times 7 is 35,... and the decimal will be lined up here: 36.71	Glena 12,2,1 5x7.000 = 35.000: 36.71. Changed to 3.671 because 3 places in 35.000	Sue 18,1,0 367.1 . 7x1 . "I'm not sure."
Jay 23,1,0 8x1=8: 36.71 The product must be larger than either factor.	Josh 18,2,0 7.0x1.0. Two places: 36.71	Bert 21,1,0 "Four places." Provoked to estimate. "7 times 1/2 is 3-1/2:" 3.671.
Lulu 30,1,0 7x5=35 : .3761 "Four decimal places"	Jo 20,0,0 7x1=7 "I don't remember."	Will 28,1,0 "About 7. Four places:" .3671
Jed 31,3,1 7.000x1.0=7.000 3.500 "About half of 7:" 3.671	Eva 28,2,0 7x1=7 "Closer to 3 than to 36." Less than 7 because .5 < 1.	Wanda 31,1,0 "About half of seven."

*Student Pseudonym, Total Score (max=33), Place the Decimal Point Score (max=3), Score on Item 17 on written test.

force is the rule to count decimal places. The responses of Glena and Josh, of allowing the number of places in the estimate (zeroes at that) drive the "count rule" is the epitome of form without understanding.

TABLE 3

SELECTED INTERVIEW DATA FOR TRANSLATION
ITEM NUMBER 20

GRADE SIX	GRADE SEVEN	GRADE EIGHT
Tony 14,0,0[*] .31 2.07	Max 9,0,0 103 x207	Rhoda 15,0,0 103 x702 comp.correct
Carl 16,0,0 13 "This would be x27 13 and 27."	Glena 12,0,0 1302 "No 7000 decimals" many zeroes 9114000	Sue 18,0,0 130 270
Jed 31,2,1 0.31 2.07 217 0000 6200 0.6417	Josh 18,1,0 Written: 1.3 x2.7 comp.correct "I can't remember how."	Bert 21,1,0 .13 .72 comp.correct "Four after the decimal."

[*] Student Pseudonym, Total Score (max=33), Translation Score (max=2), Score Item 20 on the written test.

Table 3 briefly summarizes the results of the interview regarding Item 20: Multiply the number 1 hundredth, 3 tenths, by the number 2 ones, 7 hundredths. In most cases in the interview, the student was not asked to complete the computation, but just to "set it up", thus translating. It is striking that four of the five grade sixes were able to translate correctly. Jo's response, 130 x 702, characterizes the substituting of "hundred" for "hundredth" and "ten" for "tenth". Wanda, who missed the written Item 17 but corrected it in the interview, put it this way: "I always get confused on the endings...I can't remember if it's the decimal or the other one because they are so similar." This situation is not helped by the common practice of referring to 3.26 for example, as "three point two, six" rather than "three and twenty-six hundredths".

The most difficult subtest was the last. The general direction is: "Write a number which would be between the two numbers. Item 32 (3.7 and 3.71) is judged to be the best of the three for judging understanding. Of those writing the test, 31 were successful on Item 32 including 5 of the interviewees. An additional two were successful during the interview. Surprisingly in the interviews four responded 3.70. Jo explained: "Zero is lower than 3.71 and higher than that [3.7]." Three grade eights expressed on the written test or orally, "I don't know." Jed indicated any one of 3.701, 3.702, 3.703.... "I would get nine correct answers." Will used the strategy of finding the average on Item 33.

In conclusion, students can perform computational procedures for multiplication of decimals. Rows of partial products as Jed's response in Table 3 show inflexibility. There is a dependence on algorithmic rules such as "count the decimal places", even in the face of conflicting information from estimation of the product. Concepts of decimal numbers are not well understood, even by grade eight students. Recommendations include long term teaching experiments to study the processes by which students get an early grasp of the concepts before proceeding to algorithms.

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EFFECTS OF A SECONDARY MATHEMATICS METHODS COURSE ON PRESERVICE TEACHERS' VIEWS OF TEACHING MATHEMATICS

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Two levels of effect are found and discussed. The first is a practical level which provides training and experience in developing lesson plans and presenting material in a classroom situation. This level fulfills the students' expectations for the course, with the exception of coping with disciplinary problems. The second level focuses on the preservice teachers' conceptions of mathematics teaching and looks for growth during the course. Both the range of conceptions with which the students enter the course and the changes during the course vary widely. Perry's (1970) scheme is found to be a useful tool in attempting to understand these changes.

THE STUDY

During the winter of 1985 a study was begun to investigate how students in a secondary mathematics methods course viewed themselves as mathematics teachers and what effect the course had on their views. The course consisted of six weeks of on-campus instruction, followed by a four week practicum in a ninth grade advanced geometry class at a local high school. The first part of the course emphasized different ways of teaching concepts, generalizations, and problem solving. Each intern made three observations of the high school class during this period. There were 14 methods students (interns) grouped in seven pairs. Assigned to each group was a doctoral student who helped them prepare mathematically for teaching geometry, observed the geometry class with the interns, helped prepare teaching assignments, and provided feedback after each lesson taught. Each intern taught a minimum of two class periods to half the geometry class and was assigned two students from the class to monitor (including grading of homework and classwork) for the four week period. About one-half of the lessons taught were videotaped for subsequent analysis by the intern and doctoral student. Each intern was also required to keep a diary on each day's activities. The emphasis on each entry was to reflect and critique the various teacher education activities from the perspective of what the material meant to them as emerging professionals. Twenty to forty-five minute interviews were conducted during the final week of the course with each methods student. The interviews were designed to elicit student assessment of the course activities and to generate self-assessment of the student as a teacher. Follow-up interviews were conducted during the summer, subsequent to student teaching, to provide evidence of how student teaching affected their perspectives on teaching and on the methods course.

STUDENT EXPECTATIONS

The initial diary entries reflected the goals the students imparted to the course and their perception of their individual strengths and weaknesses as mathematics teachers. There was a general tone of excitement at the prospect of student teaching, mixed with a marked fear that centered on two areas: lack of knowledge of mathematical content, and not knowing the "mechanics" of teaching, particularly the preparation of lesson plans. Instruction in these mechanics is seen by the interns as the main purpose of the methods course. A typical comment was:

> This is the class where my questions on lesson plans, material on tests, and other classroom ideas are

supposed to be answered.

The most common strength mentioned was an ability to "get along with kids." Several students documented this assertion with stories of substitute teaching, working in day care operations or coaching various sports. The fear of lack of content knowledge arose in different forms which provided some insight into the students "levels" of thinking. While some of the students seemed concerned about their abilitv to "do" geometry - i.e., to make an "A" on homework and tests - others questioned their understanding of geometry at a level that will help them in the classroom. This difference in perspective, between viewing the course and its contents as merely subject matter (as one student put it: "guessing what the instructor wants") and beginning to see oneself as a teacher, was evident in both the mathematical content and the pedagogical content of the methods course.

TEACHING

The students found their first teaching experience to be generally traumatic - both in anticipation and in practice. One student stated:

> At times I felt like I was teaching to a two-dimensional picture. I really was not aware of anything or anyone other than myself.

Most interns tended to focus on the experience from an ego-centric viewpoint, stressing "performance" in a mechanistic sense (How many times did I do that?). The students' ability to be analytical about their own teaching was rare although two of the students did demonstrate some ability to reflect on their own teaching. While added experience tended to lessen the initial nervousness and facilitate their concentration on student performance, their classrooms were far from student-centered. The diaries emphasized the desirability (from the interns' perspective) of a teacher-centered classroom.

Bush (1982) found that enculturation, i.e., learning from prior experience as a student, was a primary source of pre-service teachers' knowledge of teaching techniques. Similarly, we found that the students held strong beliefs as to how they will teach, formed in part from favorable past experiences or in reaction to unfavorable ones. These views were social or mechanical in nature. They viewed their role as a teacher in terms of, for example, being a friend to the students, giving homework regularly, or giving extra credit, in a content-free context. Few had given consideration to the student's learning of mathematics or are aware of alternative methods of presenting mathematics. The methods course material, instruction on different types of lessons, different "moves" used in developing concepts, was generally regarded as only tangentially relevant to the real classroom. Several students reported that they saw some benefit in such knowledge, but were precluded from using that knowledge in student teaching because of perceived stringent curriculum requirements.

One characteristic exhibited by some interns was a tendency to map their own learning styles onto the student. For example one intern who credited all of her success in mathematics to "hard work and practice" was suprised at her ninth graders' proficiency with fractions and she commented "they must really have done a lot of those." These students, an advanced group, had actually spent less time practicing such skills than average students.

INDIVIDUAL CONCEPTIONS AND GROWTH

In addition to an overall look at the class, several more detailed case studies were prepared. Below are brief capsules of four of these.

JIM. The students were, in general, reserved in their comments.

Few strongly declarative statements were made. This student, however, had no compunctions about stating his mind. He was quick to suggest improvements in nearly every phase of the high school and college curricula. His comments concerning his peers were often caustic, to the point of being insulting. Every problem seemed to have an answer - his. For Jim, whose worldview was dualistic, the course seemed to serve only to reinforce his previous beliefs.

JANICE. This student exhibited, from the beginning, a degree of insight well above the average student. She seemed to have the ability at every stage to view what was transpiring from a "teacher" perspective. She was generally acknowledged by her peers to be the best teacher during the field experience. She was the only one to discuss the effect a teacher might have on students' conceptions of mathematics. Janice had recently graduated with a degree in management science, and readily found applications from this field during student teaching. For the other interns, any discussion of the relevance of mathematics to other fields was typically relegated to the area of developing motivational examples to begin lessons.

TOM. Along with Janice, Tom was the only student that could be considered to be in a relativistic position. He was acknowledged by his peers as having the most "style" among the members of the class; he had a dominating personality. At the beginning of the course he viewed teaching as a performance - showing little concern for his students'learning. He demonstrated, more than anyone else, a marked degree of development from student to teacher during the course. This seemed to result from his ability to perceive weaknesses in himself and to try to address those weaknesses. He accepted criticism in a constructive manner, and was able to observe characteristics in others that he viewed as possessing himself.

BARRY. This intern, early in the methods course, made the statement:

> I didn't know it took so much to prepare a lesson. Teaching isn't easy like I thought it was in high school. I thought the teacher only went by the book.

In the post-student-teaching interview he described his teaching style as "going by the book." Barry's worldview was generally regarded as multiplistic, but his reaction to student teaching was basically dualistic.

CONCLUSIONS

The role a methods course plays in a teacher's development is far from uniform. In one sense each student participated in the same course and received the practical preparation for student teaching they anticipated. But for some this was a time of significant growth in their conceptions of their role as a mathematics teacher. Others seemed to gain only mechanical teaching techniques from the course. Perry's scheme proved useful in attempting to understand this variation. Those deemed to have a more relativistic worldview seemed to profit most from the course. Few students exhibited levels of development above the multiplistic stage.

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PME-NA COLUMBUS, OHIO OCTOBER 1985

PAPER PRESENTATION ABSTRACT

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Title of Contribution: An Investigation of Students' Knowledge of Mathematical Iteration

ABSTRACT

Mathematical iteration is a process which is used in the execution of a variety of algorithms for a variety of purposes. It results in the production of a sequence in which each non-initial term is determined from its predecessors in the same way. Mathematical iteration is not a common topic in school mathematics, despite recommendations that it be included in the mathematics curriculum. A knowledge of iteration may give students a framework for understanding mathematical concepts, such as real numbers and the derivative, for exploring problems, and for carrying out procedures. As an instructional topic, mathematical iteration also provides a context within which students may develop and extend their knowledge of mathematical concepts and procedures.

The development of students' knowledge of iteration was investigated as four ninth and tenth grade students were engaged in BASIC computer programming tasks. The students developed their knowledge of iteration as they were involved in the construction, execution, revision, and refinement of computer programs. The emphasis was on the students conceptualizing and developing their own algorithms which employed iteration and then operationalizing their algorithms in computer programs.

Four students, two girls and two boys, were selected to participate in the study. One of the boys was enrolled in a second-year algebra course; the other students were all taking high school geometry. The students had very little experience with computing and had not previously written their own programs.

The treatment extended over 10 weeks and consisted of 13 group teaching sessions and 3 individual interviews for each student. The first five teaching sessions and the first set of interviews dealt with problems related to the generation of numerical sequences. The remaining sessions dealt with iterative methods for solving quadratic equations.

The sources of data included video-tapes and audio-tapes of the interviews and teaching sessions, written student productions

such as printed copies of their programs and their notes on problem solutions, and my own notes taken during the sessions and during reviews of the tapes. The process of analyzing the tapes was ongoing throughout the study. After each session, I reviewed the tapes and filled out indexing sheets to identify emerging patterns of behavior and indicators of their knowledge of iteration. The students' errors and difficulties as they developed computers programs were also noted.

Three components of a knowledge of iteration were identified: (a) specifying one or more initial values, (b) repeatedly executing a procedure to produce subsequent terms of the sequence, and (c) applying an appropriate rule for stopping execution of the procedure. These components can be identified as parts of computer programs that were studied and developed by the students during the investigation.

The students used various strategies as they operationalized their algorithms and modified and refined their programs. For example, during the sessions related to the generation of sequences, the students developed a chart first and then attempted to "teach" the computer to produce an identical chart. The students either identified a pattern in the chart and used that pattern to express the changes in the variables, or they attempted to "teach" the computer the steps that they followed when they set up the chart. These steps may or may not have involved a pattern which was identified.

Difficulties that the students had as they developed the programs were also identified. All of the programs involved the use of two or more variables whose values changed throughout the execution of the program. The use of a FOR/NEXT loop seemed to be conceptually more difficult for the students because it required them to reorder the steps in their own algorithms used to construct their tables.

During the proposed session, samples of the students' work will be presented. Indicators of the three components of their knowledge of iteration will be discussed as they were evidenced in the programming activities. Programming difficulties related to the three components will also be discussed.

METACOGNITIVE AND PERFORMANCE ASPECTS OF INCLUSIVE SOLUTIONS

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One solution is more inclusive than the other if it contains all the information in the other solution plus some additional information. Students in college algebra classes were given word problems of different inclusiveness and rated their complexity, their typicality, and their potential usefulness for solving a similar problem. Students' selection of potentially useful solutions failed to support the hypothesis that they would prefer more inclusive solutions, but they did show significant preferences for the more simple and the more typical solutions. A less-inclusive solution was not effective in helping students solve any of 6 test problems but a more-inclusive solution helped students solve 3 of the 6 test problems. The limitation of metacognition is revealed by the finding that the 3 effective solutions were the ones least often selected as being potentially useful.

Imagine that you are given two problems to solve and don't know how to solve either one. You have the opportunity to see the solution to one of the problems and have to chose the solution that will help you solve both problems. Which solution would you choose?

Although there may not be an obvious correct choice for many pairs of problems, a correct choice should exist if one problem is a special case of the other. For these kinds of choices, the solution that is more inclusive should provide the most complete information for solving both problems. One solution is more inclusive than the other, according to the definition used here, if it contains all the information needed to solve the less inclusive problem plus some additional information. Table 1 shows the problems used in the experiment, which are ordered from the least to the most inclusive in each category.

The purpose of this study was to determine how the inclusiveness of a solution influences students' choices of solutions and their ability to solve problems of different inclusiveness. These issues were studied by giving a questionnaire to students in a college algebra class. In the first part of the questionnaire students selected, for different pairs of problems, the solution they would prefer to use in order to solve both problems. They also rated the complexity and typicality of the problems. In the second part of the questionnaire they received a solution of one member of a pair and were asked to use it to solve the other member of the pair.

The results did not support the hypothesis that students would select the most inclusive solution when it was paired with a less inclusive solution. Although solution inclusiveness did not significantly influence students' selections, the complexity and typicality of solutions did have a significant effect. Students showed significant preferences for avoiding the most complex solutions, and for selecting the most typical solutions.

The hypothesis that students would select solutions of greater inclusiveness was based on the normative principle that inclusive solutions are more informative. The second part of the experiment determined whether inclusive solutions were, in fact, more useful than less inclusive solutions. For each of the six problem categories, some of the students had to solve a problem that was more inclusive than the solution. The other students solved a problem that was less inclusive than the solution. Table 2 shows the percentage of students who were able to formulate a correct equation to represent the problem. They had 3 minutes to construct the equation (Attempt 1), 2 minutes to study a detailed solution to the related problem, and an additional 3 minutes to use the solution to construct the equation (Attempt 2). The z-score measures whether the

Table 1
Problems Used in Experiment 1

Category	Problem
Area	1. A lot is 4 times as long as it is wide. What is the width of the lot if its area is 7500 square yards? $4w \times w = 7500$
	2. A side walk, 1 yard wide, surrounds a lot that is 3 times as long as it is wide. What is the width of the lot if the total area (including the side walk) is 8000 square yards? $(3w + 2) \times (w + 2) = 8000$
	3. A side walk, 2 yards wide, surrounds a lot that is 2 times as long as it is wide. What is the width of the lot if the area of the walkway is 450 square yards? $(2w + 4) \times (w + 4) - (2w \times w) = 450$
Cost	1. A group of people paid $238 to purchase tickets to a play. How many people were in the group if the tickets cost $14 each. $\$14 = \$238/n$
	2. A group of people paid $306 to purchase theater tickets. When 7 more people joined the group, the total cost was $425. How many people were in the original group if all tickets had the same price? $\$306/n = \$425/(n + 7)$
	3. A group of people paid $70 to watch a basketball game. When 8 more people joined the group the total cost was $120. How many people were in the original group if the larger group received a 20% discount? $.8 \times (\$70/n) = \$120/(n + 8)$
Distance	1. A pilot flew 1575 miles in 7 hours. What was his rate of travel? $1575 = r \times 7$
	2. A pilot flew from City A to City B in 7 hours but returned in only 6 hours by flying 50 mph faster. What was his rate of travel to City B? $r \times 7 = (r + 50) \times 6$
	3. A pilot flew his plane from Milton to Brownsville in 5 hours with a 25 mph tailwind. The return trip, against the same wind, took 1 hour longer. What was the rate of travel without any wind? $(r + 25) \times 5 = (r - 25) \times 6$

Table 1 (continued)

Category	Problem
Fulcrum	1. Laurie weighs 60 kg and is sitting 165 cm from the fulcrum of a seesaw. Bill weighs 55 kg. How far from the fulcrum must Bill sit to balance the seesaw? $60 \times 165 = 55 \times d$
	2. Tina and Wilt are sitting 4 meters apart on a seesaw. Tina weighs 65 kg, and Wilt weighs 80 kg. How far from the fulcrum must Tina sit to balance the seesaw? $65 \times d = 80 \times (4 - d)$
	3. Dan and Susie are sitting 3 meters apart on a seesaw. Mary is sitting 1 meter behind Susie. Dan weighs 70 kg, Susie weighs 25 kg and Mary weighs 20 kg. How far from the fulcrum must Susie sit to balance the seesaw? $20 \times (d + 1) + 25 \times d = 70 \times (3 - d)$
Mixture	1. An alloy of copper contains 23% pure copper. How much of it must be melted to obtain 5.3 pounds of copper. $.23 \times p = 5.3$
	2. One alloy of copper is 21% pure copper and another is 31% pure copper. How much of the 31% alloy must be melted together with 9 pounds of 21% alloy to obtain 6.7 pounds of copper? $.21 \times 9 + .31 \times p = 6.7$
	3. One alloy of copper is 20% pure copper and another is 12% pure copper. How much of each must be melted together to obtain 60 pounds of alloy containing 10.4 pounds of copper? $.20 \times p + .12 \times (60 - p) = 10.4$
Work	1. Tom can mow his lawn in 1.5 hours. How long will it take him to finish mowing his lawn if his son mowed 1/4 of it? $.67 \times h = .75$
	2. Bill can paint a room in 3 hours and Fred can paint it in 5 hours. How long will it take them if they both work together? $.33 \times h + .20 \times h = 1$
	3. An expert can complete a technical task in 2 hours but a novice requires 4 hours to do the same task. When they work together, the novice works 1 hour more than the expert. How long does each work? $.50 \times h + .25 \times (h + 1) = 1$

Table 2

Effect of Solution Inclusiveness on the Successful Use of Solutions

Category	Solution Inclusiveness	Percent Correct		Z - Score
		Attempt 1	Attempt 2	
Area	Intermediate	0	0	0
	High	0	11	1.73
Cost	Intermediate	0	4	1.00
	High	7	47	3.20*
Distance	Intermediate	3	20	2.24
	High	13	43	3.00*
Fulcrum	Intermediate	0	10	1.73
	High	10	57	3.74*
Mixture	Intermediate	0	0	0
	High	20	20	0
Work	Intermediate	0	3	1.00
	High	0	11	1.73

* Significant at the $p < .01$ level

second attempt (when students had a solution) was significantly better than the first attempt. A $p < .01$ level of significance is used because of multiple comparisons.

The results indicate that the solution of a less inclusive problem (at the intermediate level of inclusiveness) did not help students solve a more inclusive problem. The improvement on the second attempt was nonsignificant for all six categories. In contrast, the solution of a more inclusive problem (high level of inclusiveness) did result in a significant improvement for three of the six categories.

A closer look at the three effective solutions (the most inclusive cost, distance, and fulcrum problems) is revealing. Each of these problems was considered by students to be high in complexity and low in typicality. The consequence was that students avoided selecting these solutions in order to solve both Problems 2 and 3. Only 16 of 42 students selected the more inclusive solution for the Cost problem, only 17 of 47 students selected the more inclusive solution for the Distance problem and only 18 of 48 students selected the more inclusive solution for the Fulcrum problem. Thus students show a consistent preference for an ineffective solution (Problem 2) over an effective solution (Problem 3).

ON CONCEPTUALIZING RESEARCH METHODOLOGIES FOR COMPUTER-BASED INSTRUCTION

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Current research methodologies may be inadequate to significantly enhance our understanding of computer-based instruction (CBI). Inconsistent findings and methodological shortcomings in media research, including CBI, raise this possibility. This paper suggests that considering fundamental issues in educational research may help researchers conceptualize more productive research methodologies for CBI. An examination of these issues should lead to specific recommendations for conceptualizing and conducting experiments involving the computer in instruction. Specific recommendations are discussed in the following categories: moving towards theory building, expanding research methodologies, and setting priorities for applied research.

On Conceptualizing Research Methodologies for Computer-Based Instruction

The pitfalls of educational research are legion. The risks of succumbing to these pitfalls may be amplified when research involves innovative technologies like the computer. The exciting potential of the computer for enhancing instruction generates spontaneous enthusiasm and a concomitant need to justify this enthusiasm. Consequently, there may be a temptation to short-circuit the research process by asking superficial questions, by sacrificing sound methodology and by over-generalizing findings. The history of research investigating a variety of instructional media also suggests caution for those interested in researching topics in computer-based instruction (CBI). Several writers have chronicled the failure of researchers to create a useful research base for guiding the selection and use of instructional media (Clark and Bovy, 1983; DiVesta, 1975; Jamison, Suppes, and Wells, 1974; Leifer, 1976; Oettinger and Zapol, 1971; Saettler, 1968).

Existing CBI research appears to fare no better. Evidence that methodology, for example, is a serious concern can be found in Kulik's (1983) meta-analysis of CBI studies in which a majority of the available studies were eliminated due to methodological shortcomings. Before we accept the possibility that strong empirical evidence is not possible or alternatively continue to muddy the empirical waters, we should compare the goodness of fit between research methodology and the types of questions for which we seek answers. The purpose of this paper is to highlight a set of conceptual and methodological considerations which may be relevant to such a comparison.

Fundamental Conceptual and Methodological Concerns

Research methodology is presumably a function of the types of questions the researcher chooses to address. A conceptualization of a research study begins, therefore, with a question in the mind of the researcher. The old saw about getting a good answer only if there is a good question is important at this stage but the process is more complex when put into the context of educational research. The researcher must also consider potential answers to the question, how those answers might be explained, and use these notions to formulate a methodology for research. The challenge of research is not simply generating good questions, but evolving strategies for generating a limited set of answers.

Put another way, the fundamental issue is whether or not research methodology will permit what Platt (1968) has termed strong inferences as opposed to weak generalizations. Strong inferences are, of course, preferred but are similarly more difficult to ferret out. They demand careful control of variables which sometimes means a movement towards "laboratory" as opposed to "real world" conditions. Achieving strong inferences is also facilitated by the presence of a guiding theory. A theory aids in the generation of testable hypotheses, expedites methodological decisions, and also serves as a bench mark for interpreting the data gathered. A theory enables experimental results to be interpretted as "a case in point" as opposed to an isolated phenomenon with many alternative explanations. Without a theoretical perspective isolating a set of significant variables for study becomes difficult.

As an example consider the following question which is typical of those motivating existing CBI research: Can a computer help students learn concepts in biology? Without theoretical guidance or a concern for the level at which findings might be generalized a conceptually simple experiment may emerge from this question. A biological concept is selected and taught _via_ a computer to an appropriate population after which some achievement measure is employed to compare these subjects to others taught the same concept _via_ alternative media. The best a researcher can do using this methodology is to report results and speculate broadly as to what may have caused them. At worst, a misguided or over-enthusiastic researcher will on occasion use the resulting data to make general statements concerning the usefulness of computers to teach biological concepts.

Raw empiricism is the dominant characteristic of this type of study and many of the benefits of experimental research are lost or risk being perverted. Very little is contributed to an overall understanding of the computer in instruction or for that matter its utility in teaching biological concepts beyond the specific conditions of the experiment. Methodology in this case has been conceived as only a logical extension of the question without considering the realm of possible answers. Current conceptualizations of CBI research methodology may suffer from this fundamental weakness.

Specific Conceptual and Methodological Recommendations

Even a casual review of CBI research indicates a tendency to duplicate the methodologies of other media research with little reason to believe that results will be any more enlightening. One way to address this concern is to re-examine the conceptual and methodological foundations of research involving instructional media. To be valuable, however, this exercise must result in specific recommendations which can guide the researcher. Below is an attempt to move in this direction. Recommendations have been grouped into three broad categories: moving towards theory building, expanding research methodologies, and setting priorities for applied research.

Moving Towards Theory Building

Lachenmeyer (1970) has argued that the predominant view of experimentation in the social sciences has been inefficient in that it does not facilitate the formulation of general theories. The typical research study in psychology and education investigates isolated hypotheses which are generated primarily from a review of previous research as opposed to observable phenomena. Presumably, the cumulative effect of many experiments will be the development of a general theory which consolidates findings into a unified whole. This has rarely, if ever, occurred, however, in educational research.

Most successful theory building, on the other hand, occurs when general principles are inducted from directly observable, naturally occurring phenomena. The initial goal of the researcher is to develop adequate measurement instruments and methodologies for studying a readily observable phenomenon. Later, after fact finding pilot studies, a theoretical statement may emerge to explain facts related to the phenomena studied. At this point, the theoretical statement can be used deductively to generate hypotheses for experimental verification.

The need for developing a theoretical orientation to media research has been articulated by a number of writers. Salomon (1979) and Salomon and Clark (1977), for example, have attributed the lack of consistent findings in media research to the absence of a theoretical orientation. They feel that most of the existing research is a result, in their words, of investigations with media to determine instructional effectiveness. Instead, they propose that a theoretical oreintation leads to research on media to determine psychological effects. This puts the investigation of instructional media like the computer within the realm of psychological and educational theory instead of existing in a theoretical vacuum. Similarly, Clark and Bovy (1983) have cautioned against the confounding of technology and instructional method in formulating and conducting research involving instructional media. Ellis (1976) presaged these positions when he argued that understanding the use of computers in education really begins with an understanding of education.

A move towards a theory building orientation in CBI research would have significant impact on research methodologies. More studies would be conceived, carried out, and interpreted as pilot or exploratory studies. The importance of studies which focus on sharpening measurement instruments and developing workable methodologies would be recognized. In addition, the likelihood that findings would be over-generalized would be reduced. In short, even the most applied research would be judged in terms of its contribution to a broader understanding of the computer's role as a medium of instruction.

At the same time focusing on theory building would provide a rationale for basic research in CBI. Basic research, for example, is possible when the computer is integrated into a more general theory of instructional media. Theoretical positions like those developed by Salomon (1979) and Olson (1976) could be used to guide the conceptualization of basic research hypotheses and the development of suitable methodologies. By carrying out basic research founded on accepted theoretical positions researchers would be forced to confront and tease out the significant attributes of the computer as a medium of instruction. We would see fewer studies which make recommendations as to whether the computer is a viable medium for instruction and perhaps more that would give us insight into when, where, and with whom the computer might best be suited for particular categories of instructional content.

Expanding Research Methodologies

Building theory implies a search for those variables which may account for or predict the occurrence of a range of phenomena. When dealing with human behavior and learning, the researcher is assured that the number of variables and their interactions will be enormously complex. While a theory will reduce the number of variables to consider relevant for study, considerable complexity remains. When investigating instructional uses of the computer, identifying relevant variables becomes even more difficult. Computer technology makes the options for delivering instruction so open ended that even the set of possible variables may not be intuitively obvious (Reinking, 1984).

What does this imply specifically for research involving computers in instruction? In general this means expanding methodology to recognize a wider range of variables and to permit those which are significant to surface. Methodologies which recognize and pursue only technological differences may be too simplistic. Few useful inferences can be made about the computer's relationship to instruction when so many potentially significant variables are ignored.

A desire to contend with more variables may dictate the need for more complex methodologies in experimental research. A wider range of statistical techniques may be necessary; multivariate and regression analyses may need to supplement univariate ANOVAs. Even simple designs and straightforward statistical analyses, however, can be made more powerful by including multiple dependent measures and attempting to replicate results. The latter option will, of course, require the cooperation of journal editors. The experimental bias of educational and psychological research may also need to be tempered in order to encourage non-experimental methods employing correlational analyses or even case studies. Caution may also be necessary in foreclosing areas of inquiry on the basis of accepting a single null hypothesis.

Examples of how creative research methodologies could proceed from theoretical underpinnings and a concern for relevant variables can be found in Salomon and Clark's (1977) examination of research methodologies for media research. They have offered several viable research designs and statistical techniques which would be directly applicable to CBI research.

More productive methodologies may also be a result of collaborative research which brings together colleagues of varying expertise. The interacting variables which operate simultaneously when using the computer in instruction can rarely be seen clearly by one individual. Ideally expertise would be sought out to insure knowledge of media, the instructional content and its pedagogy, and cognitive psychology. Even these areas do not exhaust all of the sources of significant variation (eg. social and environmental factors).

Setting Priorities for Applied Research

A perhaps healthy tension has always existed between basic and applied research. Under ideal circumstances these research emphases complement each other so that both theory and practice are mutually enriched. Under less than ideal circumstances the distinction between the two is unclear and research proceeds haphazardly with little clear direction. One of the theses of this paper is that the latter condition is more characteristic of media research in general and CBI research in particular.

A common misconception, however, is that basic research is uniformly more essential to the theory building process. Lachenmeyer (1970) has cited several examples from the history of science which suggests that the opposite is more likely the case. Theory building is at first more often a result of observing events in the real world which is the domain of the applied researcher. The applied researcher, however, must recognize the unique contribution applied research can make to theory building and conceptualize methodologies accordingly.

This means more than suggesting theoretical explanations for findings; it also means seeking out real instructional problems that may contribute to a broader theoretical understanding of CBI. Wilkinson (1984) has suggested priorities for selecting research topics to explore computer applications in the teaching of a content area. First, the computer should be used to manipulate content in a fashion which is not readily duplicated by other instructional media. Secondly, the use of the computer should be guided by accepted pedagogical principles for teaching the content selected. Finally, a high priority should be given to research which addresses areas of instruction which have proven to be problematic.

Researchers who subscribe to these priorities will increase the likelihood that their research will be more relevant to the practitioner as well as contributing to the theory building process.

Conclusion

In summary, current research methodologies may be inadequate to significantly enhance our understanding of CBI. Inconsistent findings in media research, including CBI, raise this possibility. Considering fundamental issues in educational research suggests a broader range of research methodologies may be appropriate. The goal of CBI research should probably be identifying significant variables in an effort to build theoretical perspectives. Finally, the topics of applied research may need to be evaluated closely to insure their usefulness for both practice and theory development.

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SOME AREAS OF NEUROPSYCHOLOGICAL RESEARCH WITH POSSIBLE IMPLICATIONS FOR MATHEMATICS EDUCATION

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Understanding the functioning of the brain has tremendous implications for both teaching and learning, yet it has only been recently that brain research has begun to give us some insights into the workings of this very complex part of our anatomy. Although the neurosciences are still in their infancy, the following areas show possible promise for research by math educators in collaboration with neuropsychologists and neuroscientists. These areas include: differences in hemisphericity, processing and learning styles: learning style preferences vs. learning style strengths or weaknesses; maturational differences including growth spurts and mylenation; effects of stimulation and enriched environments; the hierarchy of the triune brain; effects of emotions; control of attention; sex differences; and other related factors.

INTRODUCTION

The brain has been called education's next frontier, and research into this area has tremendous implications for all of education (Loviglio, 1980). Within the brain lie answers to such questions as "How do people learn? Can we increase the learning of all people from infancy to old age? Why do some people seem to learn differently than others? What makes a teaching method or material effective? Can children overcome learning disabilities?" These questions are only a few of the myriad possibilities in the area of brain functioning research. Research in this area is relatively new and definite implications cannot yet be drawn for education, but some promising areas for further exploration are being mentioned. This paper will note a few of those areas and some of the research which has already been done. Certainly much remains to be done and we must beware of "jumping on bandwagons".

DIFFERENCES IN HEMISPHERICITY, PROCESSING AND LEARNING STYLES

This is probably the most publicized area of brain functioning research and many educators have begun to talk about teaching right and left brain children. Since Roger Sperry won the Nobel prize for his "split brain" research, educators have been trying to apply his results to the classroom. A caution must be noted here because the work of Sperry and others was performed on patients who had had the corpus collosum severed. In normal children, impulses pass rapidly from one side of the brain to the other through the corpus collosum and actions cannot be thought of as being performed on solely the right or left side of the brain. Current research by educators and neuropsychologists does seem to indicate, however, that different children do have different learning styles. Davidson (1983) has identified two distinct learning styles among students learning mathematics which correspond to the different processing styles of the two sides of the brain. One style is a more holistic style corresponding to the right brain functioning and the other style is more sequential, corresponding to the left brain. While the evidence strongly disputes the notion that we learn with only one side of the brain, the two hemispheres of the brain do perform in different ways (Levy, 1983). Wheatley (1977) described the right hemisphere as "thinking" in images and the left hemisphere as "thinking" in words. This may be a simple way of looking at the differences in functioning of the two hemispheres. We must be careful not to simplify this functioning too much, however. It cannot be said that math and reading are left brain subjects and art and music are right brain subjects. Both hemispheres of the brain are important to the full understanding of any subject. Research is needed to determine better ways of fully utilizing the whole brain for all children.

LEARNING STYLE PREFERENCES VS. LEARNING STYLE STRENGTHS OR WEAKNESSES

Many researchers are using paper and pencil tests to determine the learning style preference of students. Several of these tests ask students such questions as whether they would rather give a speech or sing a song, read a book or listen to music, etc. Results from these instruments are supposed to determine if students are right brained or left brained. Davidson (1983) warns that even if a student does indicate a preference for one style that style may not be the student's strongest one. It is possible that the student may even have brain damage impairing the functioning of a preferred style. Recent developments in brain imaging technologies such as electrophysiological recording, magnetoencephalographic recording and nuclear resonance may give us more information about a child's strengths and weaknesses in processing information.

MATURATIONAL DIFFERENCES

Epstein (1979) has found evidence of growth spurts in the brain which roughly correspond to the four periods of cognitive development described by Piaget. Neuron development and mylenation of the brain also progresses in stages which match those of Piaget and may account for the growth spurts (Johnson, 1982). This has implications for education in terms of the optimal time and mode of presentation of topics. Johnson (1982, p.49) has stated, "If we present a child with learning tasks prior to the mylenation of the areas needed to handle these tasks, we may be forcing the child . . . to use less appropriate neural networks. By asking the learner to perform before the appropriate area is developed, we may be causing the failure and frustration seen in many children today." She also hypothesizes that many adults may never reach the stage of formal operations because they lacked the concrete development of concepts as children. Math is an ideal subject for the development of concepts through physical manipulation of objects, but this manipulation is often neglected in favor of paper and pencil exercises. Formal operations may be possible only if the concrete operational brain growth of the previous period has been properly developed through concrete stimulation and experiences. It thus appears that their are optimal times for the development of certain skills and children of different ages will perform differently at least in part due to the different physiological makeup of the brain. MacLean (1978, p.39) has stated, "There is now abundant anatomical and behavioral evidence that if neural circuits of the brain are not brought into play at certain critical times of development, they may never be capable of functioning." Much research is necessary to determine the relationship of learning tasks and the timing of their presentation to the growth of cognitive abilities.

EFFECTS OF STIMULATION AND ENRICHED ENVIRONMENT

Stryker and Sherk found that cats raised in an environment of only vertical lines did not later respond normally to horizontal lines. Apparently they had not developed the brain structures to see horizontal lines (Esler, 1982). Greenough found that animals raised under environmentally impoverished conditions had fewer neural connections than animals in enriched conditions (Esler, 1982). Some researchers hypothesize that the Japanese have trouble pronouncing the Letter L because of lack of experience with the sound during infancy. This tends to indicate that the physical makeup of the brain is not totally determined at birth but develops according to experience. It is important for educators to determine what experiences can promote optimal growth in children. Indeed, recent experiments with rats indicate that the brain can continue to grow even into old age when give the proper stimulation.

THE HIERARCHY OF THE TRIUNE BRAIN

MacLean (1978) argues that the mind is made up of a hierarchy of three brains which reflect evolutionary stages. The reptilian, or R-complex, located in the midbrain, seems to be responsible for certain "instinctive" behaviors in humans, such as impulses and routine habits. The second formation, the limbic system, representing the old mammalian brain seems to have a role in emotions. It is the third formation, the neocortex, or the new mammalian brain, which is divided into left and right hemispheres and performs tasks involving reason, language, and other cognitive thought. Behaviors are a result of a complex interaction between all parts of the brain and cannot be understood by looking at only one part.

EFFECTS OF EMOTION

Emotions controlled by the midbrain with no apparent external cause, or emotions controlled by the forebrain such as anxiety or worry can have a great effect on the learning of mathematics. The right brain seems to play a special role in emotions. Research has shown that when the right brain is positively engaged emotionally, both sides of the brain will participate regardless of subject matter and learning will increase (Levy, 1983).

CONTROL OF ATTENTION

The reticular formation, the limbic system, and the thalamus actively select the stimuli to which a person will respond. The human brain responds to novelty. When asked to repeat the same task numerous times the brain "habituates" and does not consciously think about the task. Therefore, much of the drill and repetition in teaching mathematical operations may actually be counterproductive to learning.

SEX DIFFERENCES

Many researchers state that there are clear differences in the male and female brain (Grow and Johnson, 1983; Loviglio, 1980; Weintraub, 1981). There are no clear indications whether these differences are due to sex hormones, differences in the rate of maturation, differences in experiences or other factors, but the differences are most likely due to a combination of these causes. Researchers are agreed, however, that sex alone does not determine the makeup of the brain. Differences within each sex are greater than the differences between sexes. It would not be advisable to set up separate math classes for boys and girls, simply because there are sex differences. It is important to look at each child as an individual.

OTHER RELATED FACTORS

Other factors have been found which are related to brain functioning and may also have implications for teaching. Space does not permit discussing all of these fully. These include such factors as hand preference, allergies, use of biofeedback and levels of cognitive awareness, and the tendancy of the brain to seek patterns. Readers and researchers will note others.

We are on the precipice of exciting new discoveries about that universe which lies within the brain of each one of us. Emerging technologies are giving us the means to begin our explorations. We cannot wait until someone gives us all the answers about how this brain functions, because those answers may not come in the foreseeable future, but we must actively seek to understand how one learns and how we might best use this knowledge to teach.

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GEOMETRY, SPATIAL DEVELOPMENT AND COMPUTERS: YOUNG CHILDREN AND TRIANGLE CONCEPT DEVELOPMENT

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ABSTRACT: A research study was conducted to see if a computer program could help young children expand their pre-conceived conception of "triangles" as a figure which is equilateral and pointing straight up. The random triangle program from the geometry section of the TABS (Technology and Basic Skills) software was used. The random triangle program generates three randomly placed dots on the monitor, then the dots are connected to form a triangle. Twenty five preschool children were pretested on their conception of triangle. Twelve children had the conviction that the shapes were triangles only when they were equilateral and pointing straight up. After the children interacted with the random triangle program several of the children immediately generalized their conception of triangles to include all shapes and orientations, including a child who had just turned two-years-old. Some of the children also made connections between the triangle outlines on the screen and triangles cut out of paper.

CONNECTION BETWEEN SPATIAL ABILITY AND GEOMETRY

The importance of spatial abilities in the development of geometric understanding has been studied by many scholars (Davis and Silver, 1982; Fey, 1982). Some of the documented connections between spatial abilities and geometry are: to recognize a shape (Fuys, 1984), to see a rotated shape (Rosser, Horan, Mattson and Mazzeo, 1984), to predict what a shape will look like when rotated or flipped (Pellerey, 1984), to recognize patterns, to see symmetry, and to form a concept image (Vinner and Hershkowitz, 1980). The first two of these connections can be broken down into a somewhat flexible sequence that the author has observed young children go through on their way to understanding the concept of triangles.

MALLEABLE SEQUENCE OF TRIANGLE CONCEPT LEARNING

1. Not able to match any shapes; circle, triangle, square, etc.
2. Able to match shapes; circle, (equilateral - horizontal base) triangle, square.
3. Match shapes; oval, diamonds, rectangles.
4. Associate name of shape with those listed in # 2 and # 3.
5. Recognize that a rotated equilateral triangle would be a triangle if it were put "Right side Up", and the child will physically turn the triangle "right side up".
6. The child calls a disorientated triangle a "Funny Triangle".
7. The child recognizes disorientated triangles as triangles, no qualifiers needed.
8. Right triangle with a horizontal base recognized as another kind of triangle.
9. The child recognizes disorientated right triangles.
10. The child recognizes "other triangles" with horizontal bases.
11. The child recognizes "other triangles" when disorientated.
12. The scalene triangle with a base at a 45 degree angle is the last to be recognized.

The sequence has been called malleable because it seems to be dependent on the child's experience with shapes and shape names. The above list is flexible in that a child may have not completely learned one thing before starting to work on another. Many children seem to get stuck between steps 5 and 7. It is the author's observation that the children who get stuck have been presented with equilateral triangles (and maybe right triangles) with a horizontal bases, at the same time they were presented with "circles" and "squares". The significance of this is that circles can only vary in size and color, not in outline. The same is true of squares! Have you ever heard anyone tell a very young child that, "Triangles are strange because they do not always have the same shape."? Herbert Klausmeier's (1976) model for attainment of concepts would help identify the problem as the child failing to generalize that two or more forms of triangles are examples of the same concept. Evidence has been documented that one contributing factor is the lack of good examples and of non-examples (Dienes, 1961; Shummway and White, 1980).

Triangles are one of the shapes which are usually taught to young children. The example of triangles which is most often presented to young children is that of a solid figure with three straight sides and a horizontal base. The mathematicians who deal with the concept of teaching shapes to young children stress that the shape is the outside, not the body (Richardson, Goodman, Hartman & LePique, 1980). The idea that the real shape of the triangle is the outline isn't usually taught, and the points of the triangle are not focused on. Very little of this mathematical understanding has transcended into early childhood classrooms. Most of the teaching materials, hands on materials and examples in books, only include the equilateral triangles and usually with a horizontal base. Once in a while an isosceles triangle will be included, and very rarely will a right triangle or a scalene triangle be presented to young children.

One of the reasons for this discrepancy between research and practice in early childhood centers could lie in the fact that teaching points and lines between points is harder to do, and teachers have a hard time of generating enough samples for children to interact with and grasp the concept. One solution for this problem may be the computer. The technology of computers makes it possible to generate an infinite variety of triangles in a short time, and as Dreyfus (1984) pointed out - the computer is an appropriate tool to use for teaching mathematical concepts.

The TABS (Technology And Basic Skills) Exploring Triangles disk was conceptualized and developed to help 4th and 5th graders to intuitively understand that triangles do in fact come in different shapes and orientations. A draft version of the TABS Exploring Triangle disk had two interactive programs which could generate many different examples of triangles. The first program was "Random Triangles", and it is the program that was used in the reported study. In this activity the child pushes any key and three dots randomly appear on the screen. The child can look at the dots for as long as he or she wants, and can guess whether or not there will be a triangle when the dots are connected, and can then push any key and the connecting lines will be drawn.

STUDY

The purpose of this study was to check out the hypothesis that the "Random Triangle" program could help young children grasp a concept that usually is beyond their understanding because it is presented in abstract ways with few examples. Children often develop misconceptions of triangles which they keep for many years. It was reasoned that young children have no trouble learning complex concepts like dog, so why do some children at all ages have trouble with the concept of triangles? Mitchelmore (1984) reported that even though young children are aware of the shapes in their environment, most of them are not presented with the concepts in a way they can understand.

PROCEDURE

1. Sample: Twenty five children were pre tested for knowing the concept of triangles, and triangles other than an equilateral triangle pointing straight up. The children were from middle class families and attended a day care center. The children ranged in age from 21 months to 5 years, 7 months.

2. Pretest: The twenty five children were shown various shapes and asked if they were triangles or not. The shapes were cut from red construction paper. The set included 7 assorted triangles, and 5 non-triangles which were rectangles or other shapes with straight sides. Seven children recognized all of the triangles in all orientations on the pretest. Three of the children (all less than 2 1/2 years old) did not recognize any shapes. Twelve of the children tested were at the stage where they recognized triangles only if they were equilateral and pointing up, or said that it would be a triangle if it were turned "right side up". The eighteen children who did not recognize all of the triangles in all orientations became the treatment group.

Some of the children who only recognized the equilateral triangles with a horizontal base had names for some of the other triangles. The right triangle from the unit blocks was often called a "ramp", and was usually described with accompanying arm movements of driving a car up a ramp, and often with sound effects. One girl identified a tall isosceles triangle as "on the church", a boy called the same triangle a "tent".

3. Treatment: Participation by the children was voluntary on their part, during their free play time at the center. The children interacted with the program from 2 to 4 times each. The children could stay with the "game" as long as they wished. The children used the TABS Random Triangle program on an Apple IIe clone. The author sat by the computer and assisted the children when they needed help. The author helped the children make a "game" out of guessing whether or not there would be a triangle when the dots were connected. This "game" was played by the child pushing any key on the keyboard to get three random dots on the screen. The dots were counted, the child guessed whether or not it would be a triangle when the dots were connected, and then

the child pushed a key on the computer key board to find out. If the child guessed wrong, the "computer tricked him/her." Some of the children were asked to describe what the triangle would look like; tall, short, fat, big, little, etc.

The children were encouraged to make up their own games. They often counted the dots, including a two-year-old who did not have one-to-one correspondence yet (but got the intuitive concept image of triangles from this program). This two year old led the way for a several of the older children. He would occasionally stop with a triangle on the screen, and then hunt through the red paper shapes from the pretest and pick out the one that matched the triangle on the screen. He would then put the paper triangle on the screen and rotate it until the screen triangle was covered up. Sometimes he would say things like "all gone now", other times he just smiled.

4. Post test: Several days after last interacting with the computer and the adult, the children were given the same set of cut shapes that were used on the pretest, and asked to tell if they were triangles or not. The shapes were presented in a variety of orientations to test for level of concept.

5. Results: The children averaged the same length of time to play with the "Random Triangle" program as they did with software which was designed for preschoolers. That average time was 9 minutes per game. The actual times ranged from 45 seconds to 1 hour and 25 minutes!

Eight of the children could identify all of the triangles (in any orientation) and all of the non-triangles on the post test. Three children made progress by moving from one stage to another, but not reaching the stage of recognizing all triangles in all orientations.

CONCLUSIONS

Young children can intuitively understand the broad concept of triangles. The computer can help the young child acquire the concept of triangle by producing an infinite variety of examples. The interaction and use of the TABS Random Triangle program was fun for the young children and helped many of them form a broader concept of triangles.

Concept learning research is needed on comparing computers to the other materials that are usually available in early childhood classrooms. Materials with which large numbers of examples and non-examples can be generated are things like; geoboards, tinker toys, clay, and the sand box. Research is needed which would help the preschool and kindergarten teacher know how and when to teach concepts and with standard materials and/or with the computer.

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MATHEMATICS LEARNING AND AUDITORY PERCEPTION

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A three-phase assessment procedure was implemented to identify possible areas of dysfunction in auditory-visual perception and processing by mathematics-impaired subjects. This assessment included testing of peripheral acuity, central auditory processing, and cortical integration. Assessment results indicate specific difficulties with memory for sequence, sound mimicry, and the ability to recognize symbols presented orally.

INTRODUCTION

The difficulties that a child has learning mathematics have commonly been laid to such 'causes' as lack of motivation and inattentiveness, with the inference made that a child is a slow learner or is 'learning disabled'. Experience with children enrolled in The University of Toledo's Mathematics Clinic has led us to question the appropriateness of those labels. Although each of the children had experienced difficulty learning mathematics, many demonstrated instances of deep mathematical insight.

The purpose of this investigation was to determine if there were any auditory reception, auditory conduction, or auditory processing difficulties among the children studied that may account for any of the mathematics learning difficulties. Cathcart (1974) identified listening ability as being the most significant non-mathematical variable but did not define what he meant by the term 'listening ability'; although he did suggest that future researchers investigate general attentiveness as an independent variable with respect to mathematical achievement. Sawada and Jarman (1978) examined the relationship between mathematics achievement and specific cross modality functions. They determined that a relationship does exist and that it varies with the IQ of the student.

MATHEMATICS TEACHING

Fuson and Hall (1983) demonstrate that most children bring a wealth of information to school. In the first grade, this background is reinforced with the focus placed upon learning to count, basic addition, and basic subtraction. In the early primary grades the mode of instruction is to a large extent iconic and somewhat tactile; however, by the end of second grade the dominant mode becomes verbal/graphemic. The child is able to rely upon pictoral representations, counting, drawing directly from visual representations to develop the abstractions necessary for understanding mathematics. By the end of second grade, algorithms such as addition and subtraction with regrouping are first taught. Hiebert (1984) terms this "site 2" and defines it as the stage where "form and understanding are linked when children connect a procedure or algorithm with the underlying concept or rationale that motivates the procedure".

Because mathematical algorithms are commonly presented as series of oral/visual instructions to be memorized, an adult or child with below normal auditory/visual perceptual abilities may have difficulty understanding these instructions. That person may view the instructions as a series of meaningless words or phrases and thus be unable to transcode the verbally received instructions into graphemic representation. The child may also have trouble moving from the graphemically presented stimulus, e.g., blackboard demonstration, to an oral response.

It is possible for a child or an adult with an auditory/visual perception impairment such as those described in this presentation to acquire a good understanding of mathematics taught in the primary grades, e.g., basic addition, subtraction, etc., yet experience difficulty learning mathematics in the middle school classroom. This can occur because the nature of the content to be learned in the early grades requires an understanding of a limited number of graphemic symbols which can be acquired visually or tactually by the student without resorting to complex auditory processing. However, understanding mathematics in the upper grades requires that the student be effective perceptual processor in both the visual and acoustic symbol systems. If the transcoding process is inefficient and requires additional processing time, the student places heavy demands upon his auditory memory system with the result being a loss of information prior to completion of problem solving tasks.

METHOD

A three-phase assessment procedure was established and implemented to identify possible areas of auditory dysfunction and to identify contrasting normal and abnormal performance patterns. Phase I of the study determined the status of the peripheral auditory system. This was accomplished by obtaining a pure-tone air conduction and bone conduction audiogram. In addition, Speech Reception Thresholds (SRT) were obtained for both ears using the Central Institute of the Deaf (CID) spondaic word lists. Speech Discrimination Percentages (SDP) were obtained at the 34dB sensation level. All speech stimuli were presented from tape to the child.

In Phase II, a battery of tests shows functional ability of the auditory system from the inner ear to the primary auditory reception area in the temporal lobe of the brain. Speech signals were distorted in ways that made speech more difficult to decode. The methods included filtering the speech, providing competing signals, and providing alternating signals, thereby placing gradually increasing burdens upon the central auditory pathways. Individuals with normally functioning central auditory pathways are capable of correctly decoding the messages. Those individuals who have impairments in the conductive system will show decreased accuracy in decoding the test messages.

Phase III consisted of administering the Goldman-Fristoe-Woodcock battery of auditory skills (GFW) (1970) for the purpose of identifying possible dysfunction in the auditory/visual perception system. The series consists of twelve subtests which are divided into two screening subtests, three auditory subtests, and seven sound-symbol processing subtests. These tests and their tasks are as follows: 1.) Selection attention; 2.) Recognition memory; 3.) Memory for content; 4.) Memory for sequence; 5.) Sound mimicry; 6.) Sound recognition; 7.) Sound analysis; 8.) Sound blending; 9.) Sound-symbol association; 10.) Reading of symbols; 11.) Spelling of sounds.

Nine subjects, aged 9 to 15, (6 males, 3 females) enrolled in the University of Toledo's Educational Improvement Center were examined. Each child had experienced difficulty learning mathematics but none qualified for Special Education. None had been diagnosed as having 'hard' signs of neurological impairment.

Table 1

GFW Percentile Scores, Key-Math Grade Equivalencies

Students	SB	JD	RG	SL	RM	AM	BN	ED	KW
Sel Attention	6	8	18	1	25	1	15	1	1
Diag Discrimination	95	1	95	95	45	12	64	65	75
Recog Memory	9	26	62	30	78	18	5	35	5
Mem Content	1	1	38	12	95	58	25	35	62
Mem Sequence	1	16	24	8	62	12	18	25	35
Sound Mimicry	1	22	22	95	30	5	12	10	12
Sound Recognition	60	62	58	18	65	25	75	29	18
Sound Analysis	12	80	32	16	48	65	22	5	1
Sound Blending	25	68	26	26	25	28	48	18	18
Sound-Sym. Assoc	12	80	99	55	1	1	1	5	35
Reading Symbols	32	42	80	24	55	80	60	35	20
Spelling Sounds	32	70	28	5	78	38	32	1	1
Key Math Level	4.3	3.7	5.0	3.3	6.2	2.5	3.8	3.4	5.6
Grade Level	6.6	6.0	8.7	5.9	8.0	3.1	8.0	4.0	8.0
Age	14	10	15	13	13	9	10	10	13

The results of our study are as follows:

Phase I assessment revealed that all subjects had normal peripheral hearing acuity. Phase II assessment revealed that all subjects performed in the normal range. Table 1 presents percentile results of Phase III testing. The results indicate that all subjects were classified as having abnormal auditory perception performance.

Because of age-related ceiling effects, the results for the categories of Selective Attention, Diagnostic Discrimination, Recognition Memory, and Sound Recognition need further examination. With one exception, all students scored below the 50th percentile in their ability to recall sequences. Of particular importance is Sound-Symbol Association, the ability to identify named symbols. Six of the students scored well below the 50th percentile level in that category.

REVIEW AND RECOMMENDATIONS

Some of these math-impaired students are better able to process visual symbols than auditory symbols, while others reverse the process. We should be prepared to design our instruction to emphasize the area of most efficient processing. Additional testing is needed to determine if there is a set of standard auditory perceptual profiles that can be related to specific types of math breakdown. Our preliminary work with individuals who have been enrolled in the Auditory Perception Clinic and the Mathematics Clinic indicates that utilization of test results in designing mathematics remediation has been fruitful.

Rehabilitation attempts must take into account each individual's strength and weaknesses when designing a program for that student. If it can be demonstrated that a child has difficulty understanding a series of spoken instructions, and has difficulty attending to simple listening tasks, it should be readily apparent why the child has difficulty understanding instruction which is primarily oral.

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DIAGRAM DRAWING: EFFECT ON THE CONCEPTUAL FOCUS OF NOVICE PROBLEM SOLVERS

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Novice problem solvers seem to focus more on choosing procedures and less on the concepts involved than do expert problem solvers. A study was undertaken to determine whether novice problem solvers would focus more on concepts if they were encouraged to solve problems using diagrams. The study examined pencil and paper work of ten subjects on a problem involving fractions. The pencil and paper work was followed by a clinical interview with each subject. During the interview each subject was asked to explain her written solutions and then was asked to solve the problem using a diagram. Although the problem required a two step solution, subjects chose one arithmetic operation. Diagram work resulted in greater success, more conceptual involvement, and more confidence in the solutions.

Landau (1984) hypothesized that imaging can offer the student the opportunity for a conceptual approach rather than a procedural approach to mathematical problems. She defined conceptual as "how should I think about this problem?" and procedural as "what should I do next?". She also cited data from the Applied Problem Solving Project (Lesh 1983) that indicated that a conceptual approach was more often associated with a successful solution than a procedural approach.

The National Assessment of Educational Progress (NAEP 1979) demonstrated that American students score much higher on routine problems that require a single step than they do on multistep problems. A possible explanation for this phenomenon is that the single step problems are often solvable using a procedural approach (i.e., "should I add, subtract, multiply, or divide?"), while such an approach is inadequate for the solution of multistep problems.

In the Fall of 1984, the exploratory phase of a study on diagram drawing for mathematics problem solving was begun. The subject population was novice problem solvers from the remedial course Math 010 of the University of

Massachusetts. Part of the data from this study was analyzed to see whether novice problem solvers would focus more on concepts and less on procedures if they were encouraged to solve problems using diagrammatic representations.

METHOD

Volunteers from the remedial math class were paid to attend two research sessions on problem solving. In the first session students were given a set of problems to work and asked to show all of their work. Students returned one week later for a videotaped clinical interview in which they were asked to explain their work from the week before and to attempt diagram solutions for problems where no diagram solution had been attempted. For this study, the subjects' work on Problem #2 was analyzed.

> Problem #2: Chan has 3/4 of a gallon of ice cream. He gives 2/3 of what he has to Barry. How much ice cream does he have left?

The work of ten subjects was involved in the study.

RESULTS

Of the ten students who participated in this study, only three successfully solved Problem #2 on the pencil and paper test (first session). Of these three, two spontaneously used diagrams to solve the problem. Only one of the ten students successfully solved the problem using a straight arithmetic (symbol manipulative) approach. She (Eileen) solved the problem: 2/3 of 3/4 = 1/2, 3/4 - 1/2 = 1/4 gal. Of the seven incorrect written solutions, all attempted to do the problem in one step. Four chose multiplication, two chose subtraction, and one chose division. The work of three of the subjects is briefly described below. Their responses to both the written work and to the clinical interview characterize the full range of responses received in the study.

MARK solved the problem originally, 3/4 ÷ 2/3 = 1/12. When asked why he divided he responded, "Maybe it's subtraction...No, definitely division." His explanation suggested that he was choosing division because the ice cream was being "divided up". When asked to solve the problem using a diagram, he quickly solved the problem correctly, accurately indicating the areas of the rectangle which corresponded to the different amounts. He made no reference to any of the four arithmetic operations.

He was then asked if he could now solve the problem without the diagram. He responded, "It's subtraction not division, because division's wrong." He was taking for granted that his diagram solution was correct. He explained, "I can see it out."

MARILYN solved the original problem, 3/4 x 2/3 = 6/12. She explained her work, "I know that you are supposed to multiply and that will give you the answer." When asked to draw a diagram, she drew an appropriate rectangular area representation, indicated the appropriate regions, but was unable to name the fraction that Chan had left. She could not decide whether it was 1/4 of a gallon or 1/4 of 3/4 of a gallon. In the latter case she was using 'of' to mean "out of"; the 1/4 was out of the original 3/4.

Marilyn made no reference to arithmetic operations while working on her diagram solution. She seemed to be figuring out how the ice cream had been parcelled out.

MELISSA also originally solved the problem, 3/4 x 2/3 = 1/2. She explained that "'of' means to multiply." When asked to attempt a diagram solution of the problem, she first wanted to draw 2/3 and 3/4 separately but realized that she had no idea how to show that they are multiplied together. She settled on drawing a rectangle with 3/4 shaded. She realized that she needed to find 2/3 of that amount, but did not know how. She was stuck at that point until asked to represent 1/2 of 3/4. She thought about it, and then announced that she knew how to draw 2/3 of 3/4, which she did. Like Marilyn, Melissa was unable to name the fraction that Cnan had left. She failed to divide up the unshaded 1/4 and confused the notions of a part of a whole and a part of a part.

OBSERVATIONS AND DISCUSSION

Several trends were observed in the pencil and paper (non-diagram) solutions. Students chose one step, arithmetic algorithms to solve the problems. They explained their work with statements like "It's a take-away problem" or "of means times." When questioned about their confidence in the solution, they gave the impression that it ought to be right because they could not think of another operation that seemed more likely; a process of elimination explanation. All of these trends suggest that the students were taking a procedural approach, trying to recognize or remember the appropriate algorithm. Such an approach is consistent with instruction that focuses on key words and textbooks which feature predominantly one step, routine word problems.

Students approached the diagram solutions differently. They seemed to think about what was happening in the problem and how to represent it. They often did not know what they were going to do after the step that they were working on (an exploratory approach). They used what they knew and understood, rather than what they had been shown. Those who solved the problem felt confident in their answers, "You can just see it!" and rejected algorithmic solutions which yielded different answers.

Mark used understandings of fractions in his diagram solution that were unavailable to him when he was searching for an algorithm. Melissa and Marilyn also used understandings of fractions, but got stuck at the point where their conceptual understanding broke down. It seems desirable that their conceptual difficulties were highlighted by their work. In the algorithmic solutions, these students were not even aware of difficulties.

CONCLUSIONS

The work described here is very preliminary. The two cases (the written solutions and the diagram solutions) were not completely parallel since the first session's work was done with no questioning by the researcher, while the diagram work was done in an interview. The subjects were, however, asked to explain their original work and they generally stuck to their original solutions.

This study does however point out the procedural focus of novice problem solvers. It also suggests that involving these novice problem solvers in diagram solutions may be a valuable way to get students to work conceptually. It offers them a medium in which they can build on what they know and "can see." The increased understanding may result in a greater sense that they can determine the correctness of their own solutions. Lessons would be necessary to create bridges from the understandings gained from the diagram work to the symbol manipulative approach. This would insure that the latter approach also would be based on conceptual understanding.

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AN ETHNOGRAPHIC STUDY OF MATHEMATICS ANXIETY AMONG PRE-SERVICE ELEMENTARY TEACHERS

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This exploratory study investigated the mathematical instructional milieu of prospective teachers with high and low mathematics anxiety to determine if there are behavior differences. Two high and two low anxious students were audio- and videotaped in their mathematics instruction with small groups of children. The domains of teacher Gestalt, responses to learners' needs, verbal feedback, peer interaction, mathematical language, and personalization of instruction are discussed. Subjects' comfort in the teacher role seems to have more impact on teaching behaviors than does level of anxiety.

INTRODUCTION

Articles decrying the mathematics anxiety of elementary school teachers are common. For example, Bulmahn and Young (1982) discuss the mathematics anxiety of elementary school teachers as a communicable disease. The effect is thought to operate in two ways. First, it is presumed that positive teacher attitudes influence the development of positive attitudes in children (Johnson, 1981; Larson, 1983), and the development of positive student attitudes is one goal of teaching mathematics (Donady & Tobias, 1977; NCTM, 1980; Reyes, 1980). Second, many presume that negative teacher attitudes effect students through poor mathematics teaching, which may result in reduced student achievement (Early, 1970; Pearson, 1980; Phillips, 1973).

These two points of view have little empirical foundation. Few data are available which document the level of anxiety of elementary teachers (NCTM, 1982; Widmer & Chavez, 1982). In fact, data available concerning teachers' overall attitudes show that they have attitudes which are neutral at worst (Becker, in press; Begle, 1979; Hosticka & Traugh, 1981). In addition, evidence is mixed as to whether teacher attitudes are related to student attitudes or achievement. Although Phillips (1973) and Banks (1964) found that teacher attitudes are related to children's attitudes and achievement, Begle (1979) points out that most research studies report low correlations. Causal relationships remain undocumented.

Few researchers have investigated the instructional effectiveness of HI and LO mathematics anxious teachers. Teague and Austin-Martin (1981) found pre-service teachers' teaching performances positively correlated with their overall attitudes about mathematics, but they found no significant correlation between teaching performance and mathematics anxiety.

This study investigated the instructional behavior patterns of prospective teachers with HI and LO mathematics anxiety. They were in their sophomore spring quarter aide experience. It was their first professional education course.

METHOD

Subjects

All students enrolled in the course were administered the Anxiety subscale of the Fennema-Sherman Mathematics Attitudes Scales (Fennema & Sherman, 1976) as part of a test battery. Two HI anxious and two LO anxious were selected for observation during their field experience; all four were female.

Procedure

Each subject worked twice a week for eight weeks with two to four second or third graders. Each subject participated in two audiotaping and four videotaping sessions covering mathematical and non-mathematical content so that the mathematics focus of the research was not apparent. The taping sessions were as follows: (1) audio - nonmath, (2) video - nonmath, (3) audio - math, (4) video - nonmath, (5) video - math, (6) video - math. This schedule was designed to allow children and subjects to get accustomed to taping gradually. By the time the math videotapes were made, the equipment was a minor distraction, if any. The following content was taught by the subjects: (1) long division, (2) telling time and multiplication concept, (3) multiplication facts, and (4) subtraction with regrouping and division.

Notes were made by a GA concerning the context of the taped lessons; any unusual circumstances which might have affected the lessons were recorded. The classroom teachers gave instructions to the subjects immediately preceding each lesson as to the content of the session of which the subjects were in charge and sometimes gave them an activity with little or no instruction about how to use it. The activities and the content were both straightforward. Each subject worked with the same children throughout the series of lessons.

After all videotapes were made, the subjects were interviewed twice, first by a graduate student supervisor, then by one of the researchers. The audiotaped interviews focused on the subjects' thoughts and feelings concerning the taped lessons. The interviewers did not know which subjects were HI and LO anxious.

Analysis

The analysis was conducted using the steps outlined by Spradley (1980) including participant observation, preparing ethnographic records, interviews, domain specification, using tapes for focused observations, and conducting taxonomic analyses. Through a cyclical process of reflection, literature review, and interview tape analyses, several domains appeared to hold prospects for differences including feedback, type of instruction, use of time, affect, and use of mathematical language. Blind review of videotaped lessons (no researcher knowledge of HI and LO anxious students) yielded observable differences in several domains.

RESULTS

Teacher Gestalt

As the data were analyzed, mosaics of attributes emerged which partitioned the subjects into pairs. This partition was consistent across three dimensions of behavior; we labeled them flexibility, comfort, and empathy.

Flexibility describes an observed propensity of some students to adjust their work with children to meet changing needs. The flexible subjects handled disruptions smoothly; did more than one thing at a time; varied activities; seemed spontaneous; and showed some creativity in activities. The less flexible subjects had an authoritative manner; had difficulty keeping children on task; could not easily manage children working on different tasks; did not vary activities; expressed the need for tight control; and lacked spontaneity.

Comfort describes how at ease the subjects were in their teaching. The more comfortable pair of subjects displayed body language indicative of an open, accepting, casual manner; showed few signs of nervousness; expressed relative confidence in their abilities as teachers; and interacted frequently with the children. The less comfortable subjects seemed nervous; maintained a psychological distance from the children; expressed ineffectiveness and insecurity; and were more affected by taping.

Empathy was exemplified by sensitivity to children's needs. The more empathic subjects were accepting of children's responses; asked helping and sustaining questions; were supportive; monitored individuals; and maintained a friendly, non-threatening atmosphere. The less empathic subjects lacked awareness of when they should change approach; were brusque, condescending, or maintained a distance from the children; and talked "at" rather than "with" the children.

These descriptions of the three dimensions might seem intuitively like possible characterizations of the teaching of HI and LO math anxious teachers. However, this was NOT the case. There was one HI and one LO anxious subject in each of the pairs. Thus, one HI and one LO anxious teacher displayed Positive teaching Gestalts (PG) characterized by flexibility, comfort, and empathy, and one HI and LO anxious teacher displayed Negative teaching Gestalts (NG) characterized by inflexibility, uncomfortableness, and detachment. These attributes were consistent in both mathematics and non-mathematics lessons.

These three dimensions seemed important not only because they describe patterns of behavior, but also because differences in other behaviors seemed related to these three in combination.

Response to Learners' Needs

When researchers observed opportunities for instruction arising from students' questions or difficulties, the subject responses were coded as product (give an answer, no instruction), Socratic, didactic, or a combination of Socratic and didactic. No appreciable differences were found in use of Socratic or didactic instruction between the HI (24%) and LO (33%) anxious students. However, it was found that PG used Socratic in 46% of instructional instances, and NG in only 13%. In fact, there was an increased use of Socratic from NG (12 and 14%) to HI/PG (38%) to LO/PG (52%). Neither NG subject used any purely Socratic instruction.

Verbal Feedback

Two tapes were coded on subject responses to learner correct and incorrect answers. Responses were coded as affectively positive, negative, or neutral. NGs tend to use more positive feedback for correct responses (57% and 35%) than PGs (33% and 9%). PGs are more likely to give neutral responses than positive responses (77% vs. 23%); NGs, 55% and 44%. HIs tend to use more negative feedback for incorrect responses (17% and 18% versus 14% and 4%). HIs tend to use more positive and negative feedback on incorrect responses (18%, 18%) than LOs (11%, 10%). When all responses are combined, HIs tend to use more negative feedback (6%, 6% versus 3%, 1%).

Personalization of Instruction

Three components were examined: communications, social distance, and touching during instruction. One lesson for each subject was coded for communications data; LO/PG spoke to individuals 96% of the time, compared with 76% for LO/NG, 62% for HI/PG, and 78% for HI/NG; frequencies ranged from 190 to 240 for number of interactions with learners. The number of times the social distance decreased during a lesson by "leaning towards" was 13, LO/PG; 9, LO/NG; 35, HI/PG; and 10, HI/NG; there is a tendency for PGs to use more of this form of nonverbal communication. Since the subjects sat in various configurations and sometimes stood to work with students, a coding was made which combined touching students and their materials; touching was coded in three categories: management, positive affect for instruction or feedback, and negative affect for instruction or feedback. LO/PG tends to use touching most for positive reasons (67%) and less for management (21%). HI and NG tend to use quite a bit of touching for management (52%, 46%, 54%) and less for positive (46%, 46%, 36%).

Summary 1 -- HI and LO Anxiety

Using a criterion difference of 10%, the following statements can be made about LOs. They (1) give more neutral feedback to incorrect student responses, (2) use a greater number of precise mathematical terms (Ns = 6 and 22), and (3) have more student-student interactions (Ns = 0 and 20). Further, using a criterion of more than 5% but less than 10%, LOs (4) use more purely Socratic instruction, (5) use less touching of students and materials for management, and (6) communicate with single students more without using their names.

Summary 2 -- Positive and Negative Gestalts

Using a difference of more than 10%: (1) Positives use more Socratic instruction, isolated or in combination with didactic. Further, using a difference of 5% to 10%, Positives (2) use less purely didactic instruction, and (3) use a decrease of social distance more frequently as a form of non-verbal communication.

Summary 3 -- LO/Positive and HI/Negative

Gestalt and anxiety appear to combine effects as noted in this list, all of which represent differences of at least 10%.

1. LO/Positive uses less positive feedback to correct and incorrect responses.
2. LO/Positive uses more neutral feedback to correct and incorrect responses.
3. LO/Positive uses more Socratic instruction.
4. LO/Positive touches less for management.
5. LO/Positive touches more for positive reinforcement.
6. LO/Positive uses more correct mathematical terminology (Ns = 2 and 15).
7. LO/Positive uses a greater volume of correct mathematical terminology (fs - 18 and 110).
8. LO/Positive refers less to a single child by his/her name.
9. LO/Positive refers more to a single child without a name.
10. LO/Positive talks more to one learner than 1:2 or more.
11. LO/Positive has more student-student interaction (Ns = 13 and 0).

DISCUSSION

Again these early preservice teachers, there appear to be two significant constructs related to their instructional behaviors, (1) math anxiety, and (2) teacher Gestalt. When viewed separately, each has some impact as noted in Summaries 1 & 2. However, when viewed together, they seem to interact to produce great differences as noted in Summary 3. In the LO/Negative and HI/Positive subjects, strength in one seems to compensate for weakness in the other.

This helps account for the big differences between Summaries 1 and 2, and Summary 3.

Possibly our results were influenced by two factors which need exploration: (1) grade level and content, and (2) novice rather than expert. It is possible that because of the grade level and content (math in grades 2 and 3) that anxiety was not readily produced during the aiding sessions. It is also possible that these novices were so concerned about themselves as teachers that they were operating with concerns that took precedence over their own feelings of math anxiety. Further research with higher grades, more advanced content, and more experienced subjects is needed to clarify the roles played by these choices in the present research. In our judgement, the negative Gestalt observed in this study is of more concern in fostering poor attitudes about schooling in general than are the specific differences we found related to anxiety, especially since we observed the characterizations in non-math as well as math sessions. The anxiety-related problems noted in Summary 1 seem to lend themselves more readily to remediation than do the attributes of inflexibilty, lack of empathy, and uncomfortableness associated with the negative teaching Gestalt.

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CLASS ARITHMETIC BOOKS

Jan van den Brink

Abstract

"Class arithmetic books" is part of a larger research project which focuses on the concept of "addition and subtraction" as given during the first grade in a *realistic* and a *mechanistic* form of instruction (respectively at the Dreesschool (D) and at the Nieuwlandschool (N)).

The Dreesschool has long served as a testing ground for the IOWO (Wiskobas). The Nieuwlandschool has worked for years with the typically mechanistic arithmetic method "Niveaucursus Rekenen".

In this case-study the instruction given at the above-mentioned schools is compared, as well as their influence on student achievement and the educational ideas of both first grade instructors.

By this means, an indication is given of why the realistic arithmetic instruction is "better" than the purely mechanistic instruction.

The children from both first-grades (D and N) were divided into two evenly balanced groups: the "T-students" (test-students) and the other, non-T-students.

During the year I had regular (once or twice a week) *test-talks* with the T-students (DT and NT). With the non-T-students these talks occurred about once every two months. The idea behind this division (T and non-T) was to judge the influence of these discussions on the students.

The test talks with the students involved all sorts of first-grade arithmetic topics. Four of these were regarded in depth:

- *equations* at the D as well as the N
- *arrow language* only at the D
- class arithmetic books; at the D as well as the N.
- arithmetic stills; at the D as well as the N.

Research section: the class arithmetic book

Purpose

In march it was decided that the children would make an arithmetic book for the children entering first grade the next year.

One class arithmetic book at the D and one at the N, respectively the Drees-school arithmetic book (Dr) and the Nieuwland-school arithmetic book (N).

The class arithmetic books could be used as an instrument to recognize the various ways of instruction at the D and the N through the students'designs for the books. The children put the ideas they had gained from one year of arithmetic instruction into the class arithmetic books.

The basic question in this section of the research is: can the specific qualities of the D-instruction be found in the D-students'arithmetic book? And is this also the case for the N-instruction? And do both class arithmetic books give an indication as to research questions on:

a. introduction
b. repetition and consolidation
c. practice rows of sums
d. equations (such as $3+\bullet=10$)

The plan of the class arithmetic books

There were six assignments for the class arithmetic books which the children carried out according to various data.

Introduction and page 1 of the class arithmetic book (March 21, 1983)

"I want to make an arithmetic book of sums thought up by children. Make one page for that book. It can consist of all sorts of sums and drawings. May be we can use the book next year in the first grade when you are all in the second grade."

This assignment (page 1) was the introduction of the idea to the students. It was striking that all the children were directly motivated to make an arithmetic book for others: it gave the assignment sense.

The assignment was more or less *free*: the sums didn't have to be connected to a given period in the schoolyear.

Page 2 (April 25)

Page 2 was made a month later. The assignment was more limited than the first one. It went as follows:

"A while ago we made a page for an arithmetic book. Now we'll go further. Soon the kindergarteners will be entering 1st grade. Make all sorts of sums for them.

This assignment was primarily intended for the young children *at the beginning* of 1st grade.

Page 3 (May 30 at the D, June 6 at the N)

One month later again I asked the children to make a page for their arithmetic book with sums that the new 1st graders would do *around Christmas*.

At that time the students at the D are involved in doing arrow-sums, so it was decided to make as many kinds of arrow-sums as posible.

At the N, the students began to do "real" arithmetic (+, -, =) around Christmas, so it was decided to make as many kinds of sums as possible, often illustrated with graphic pictures.

Page 4 (May 30 at the D, June 6 at the N)

On the same day I asked the children to think up and to draw all sorts of *arithmetic games* for their book, which the new children could play. They could think up the games themselves but they had to have something to do with arithmetic.

Page 5 (June 1 at the D, June 6 at the N)

Finally, I asked the children to think up a page of sums for the last sheet in the arithmetic book.

At the D, this assignment was written on the reverse side of a worksheet. At the N, the sums were performed on graph paper because at that time the students were learning this "neat" form of notation.

Remarks

During the same week in June I also gave a drawing assignment:

"Draw yourself and a friend. You see three rabbits and your friend only sees one."

As this assignment was not given in the framework of the class arithmetic books we will not take it into consideration here.

Page 6 (June 27)

Which order?

Finally, I submitted to each child his/her three pages from May and June:

page 3 (the Christmas sums)
page 4 (the games) and
page 5 (the final sums).

I put to them the following editional problem:

"I have to make an arithmetic book, but I no longer know what order to put it in. I don't know which should go first and which after. Help me."

The children were then asked to state their arguments behind the choice of order of the three pages.

Analyses of the class arithmetic book

Both class arithmetic books were investigated for a large number of variables.

1. The category *kind*:

   Kinds of sums as to notation (arrow-language, "="-language) or as to context (graphic; substantial contexts; people, animal or object contexts; bare);

   Kinds of games (inside or outside the school; with or without chance; dramatization, etc.).

2. The category *strategic sums*

   zero sums (3 + 0 = 3; 5 - 5 = 0)

   counting sums (5 + 1 = ...; 5 - 1 = ...)

   doubling (3 + 3 = 6; 5 + 5 = 10)

   multiples of ten (20 + 10 = 30)

   positional sums (100 + 6 = 106)

   equations (10 + ● = 15)

3. The category *criticism*, such as:

   - the *order* the children chosen for the arithmetic topics and their arguments for this order.
   - *a compiled criticism* consisting of:
     - the total number of sums made
     - correct, incorrect or open sums
     - division of the signs +, - and >
     - the largest number used

These numbers were determined and the work was classified per student. For each of the four categories 1, 2 or 3 points was given. The more variety or the larger the number or amount, the higher the score. With 8 points or more, one belonged to the category "varied". Less than 8, to the category "uniform".

The report will be continued in the conference.

TABLE M

ARITHMETIC TOPICS IN THE FIRST GRADE AT THE DREESSCHOOL AND AT THE NIEUWLANDSCHOOL

	START	CHRISTMAS	APRIL
DREESSCHOOL	PLAY-ACTING AND GAMES NUMBER-LINE AND LEARNING NUMBERS (aug.) RELATIONS (>, <, =) (okt.) COUNTING of amounts acoustic counting Christmas	ARROW-LANGUAGE play-acting contexts gamrs "="-LANGUAGE sums PRACTICE thinking up sums yourself doing sums (1259 sums)	REPETITION (old topics) relations arrow-sums "="-sums number-line play-acting and games:people-animal-context acoustic counting
Nieuwlandschool	GRAPHIC ARITHMETIC drawing and colouring flannelbord pictures WRITING NUMBERS (nov.)	CALCULATINGKITE - object-contexts - learning techniques to make up sums yourself "="-LANGUAGE sums PRACTICE doing independent tasks (2150 sums) REPETITION after 10 tasks a test-task	ARITHMETIC GAMES lotto's NUMBERS (11-20) NUMBER-LINE

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ABSTRACT

ALGEBRAIC EXPRESSIONS AND PLANNING STRATEGIES: USES OF SYMBOLIC ALGEBRA COMPUTER ENVIRONMENTS TO STUDY AND TEACH THEM

Information necessary for carrying out many high level tasks in the existing curriculum in algebra and elementary functions is contained in the structure(s) of algebraic expressions. Examples are: changes of representation of functions FORMAL ALGEBRAIC <--> GRAPHIC; or strategies for comparing expressions (as in solving equations or inequalities).

The ways learners perceive the algebraic expressions appear to strongly influence their inclination to exhibit common symptoms of misunderstanding such as: their inclination to make many common algebraic errors (e.g., variations of linearity errors f(EXPR1 + EXPR2) --> f(EXPR1) + f(EXPR2)): or inability to use heuristics present in the structure of expressions to build strategies for solving equations with quadratic structure even when they demonstrate ability to use the relevant subskills such as the quadratic formula (e.g., inability to solve $2x^2y-y(1+x) = 1-x$ for x).

Protocols with high school and college students in algebra and precalculus courses were conducted. Many of these students were selected because they had made many common algebraic errors on the state-wide Junior Test in Delaware administered the the Mathematics Teaching and Learning Center. These students were given similar tasks a year later. They were also given algebraic tasks to carry out using a symbolic algebra environment developed in the Math Center. This environment was designed both for research and instructional purposes. This work was partially supported by the Greater Wilmington Development Council. Results will be reported. If time permits, the computing environment will also be demonstrated.

TEACHING REPRESENTATIONAL SCHEMES FOR THE MORE DIFFICULT ADDITION AND SUBTRACTION VERBAL PROBLEMS

Gordon B. Willis and Karen C. Fuson
Northwestern University

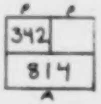
A teaching experiment was implemented to study second-graders' solutions of verbal problems containing three-digit numbers. Pictorial representations were used as intermediates in problem solution. Children were effectively taught to distinguish problem semantic type, and they improved in solution ability on several subtypes of problems.

Some types of verbal addition and subtraction problems are particularly difficult for young school-aged children (e.g., Carpenter and Moser, 1983). The present study was initiated to study such children's performance when large (three-digit) numbers were used as the givens in these problems. This was possible because these children had learned to perform single-digit computations by using finger patterns to keep track when counting on for addition and counting up for subtraction (Fuson, in press), and had then learned the algorithms for solution of multi-digit addition and subtraction in the second grade (Fuson, 1985). A second focus of the study was the implementation of a program intended to teach the abstract representation of these problems. The teaching made use of pictorial representations drawn by children as intermediates between the given problem and an arithmetic solution strategy. These representations modelled the action or state represented by the semantic content of the story, and thus distinguished problems by semantic type. Three-digit problems were a focus of the study because they are more representative of general mathematical problem solving than are single-digit problems: they require the explicit choice and execution of an arithmetic procedure rather than only a solution that is a direct model of the action in the story.

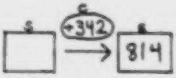
METHOD

Subjects were children from two second-grade public school classes, one (Class A) categorized by the school as containing high and the other (Class B) of average math-ability children. One pictorial representation, or picture, was developed for each of the major problem types taught. These are illustrated in Figure 1. Any of the three labelled elements of a problem may be missing, giving 12 possible problem subtypes. When given a problem, the child first identifies the major type of problem and then applies the appropriate verbal labels to the information in the problem. The child then draws the picture and fills in the known elements. He or she then uses the relationships in the picture to identify the needed arithmetic operation and writes down and carries out this operation. Addition and subtraction are therefore involved both as solution strategies, and as descriptors of the overall semantic nature of the given problem (Put-Together and Change-Get-More may be seen as additive and Change-Get-Less and Compare as subtractive in this sense). The more difficult variants of the defined problem types were selected for inclusion in a 10-item test given to children prior to and subsequent to teaching. Numbers used in problems were all three digits, and one trade (a carry or borrow) was required to obtain the sum or difference.

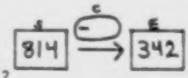
PUT-TOGETHER (COMBINE): missing PART
(All) (Part)
John and Bill have 814 toys altogether. John has 342 toys.
How many toys does Bill have?
(Part)



CHANGE-GET-MORE: missing START
(Start) (Change)
John had some toys. Then Bill gave him 342 more toys. Now
John has 814 toys. How many toys did John have to start with?
(End)



CHANGE-GET-LESS: missing CHANGE
(Start) (Change)
John had 814 toys. Then he gave some toys to Bill.
Now John has 342 toys. How many toys did John give to Bill?
(End)



COMPARE: missing BIG
(Small) (Difference)
John has 342 toys. Bill has 472 more toys than John has.
How many toys does Bill have?
(Big)

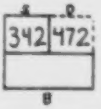


Figure 1: Examples of verbal problems and associated pictorial representational schemes with verbal labels.

All teaching was done by one of the investigators. The early teaching units each focused upon the three possible subtypes within one of the major types of problems. The order of teaching was Put-Together, Change, Compare, followed by mixed (all 12) problem subtypes. Teaching consisted of one math period in which the classification and labelling for verbal problems of a major type were described, and the complete solutions for different variants of the problem were illustrated. Then the children spent between two and four days completing practice worksheets. Little systematic individualized feedback was able to be given on the worksheets, as these were all kept for purposes of analysis. Unfortunately, teaching and testing conditions proved to be inadequate in both classes. Due to other pressures the teaching time was scattered, and insufficient time was available for practicing the mixed problem types (this was particularly true for Class B). Teaching was for each unit restricted to the current major problem type; when a new semantic type was introduced the differences between it and previous types were emphasized.

RESULTS

The variables of posttest correct picture selection and correct placement of numbers into the picture were first analyzed. For the picture-selection measure, Class A children were found to select correctly 88% of the time. The only problem type for which performance was below 85% was Put-Together, for which a Compare structure was often selected. This is not a serious error, because the pictures for Put-Together and Compare are very similar. For the measure of correct placement of the numbers into the picture, the average was 81% correct, with the lowest performance obtained on Compare type problems. The results for Class B paralleled those of Class A, but were overall about 15% lower. An analysis of the mixed-problem practice worksheets (on which all 12 possible subtypes were given) contrasted performance on the major problem types for Class A (Class B did not finish three worksheets). No differences were observed across problem type in ability to select the proper picture, but the ability to fill in the picture with the given numbers correctly did vary, with Compare problems significantly more difficult than the rest (81%, 83%, 83%, and 63% for the Put-Together, Change-Get-More, Change-Get-Less, and Compare types, respectively, $\underline{F}(3, 72) = 4.78$, $\underline{p}<.01$).

Mean correct strategy (adding or subtracting) and correct answer scores on the

10-problem pretest and posttest for both classes are provided in Table 1. For Class A, the mean strategy score rose significantly from 79% to 89% ($p<.05$). Thus, a fairly high initial level of performance improved to near ceiling. The mean correct answer scores improved somewhat more, from 56% to 72%. There was a fairly large disparity between the levels of correct strategy and correct answer in this class, mostly because many of the children often failed to write down the numbers to be added or subtracted, and as a result made computational errors. When analyzed at the level of individual problem subtype, large variations in difficulty were observed (see Table 1). Significant improvement was found on strategy scores for two problem subtypes and on answer scores for three.

Table 1
Percent Correct Performance for Correct Strategy and Answer Measures

		STRATEGY				ANSWER			
		Class A		Class B		Class A		Class B	
		pre	post	pre	post	pre	post	pre	post
PUT-TOGETHER	Missing first part	86	95	67	83	43	86*	33	67*
	Missing second part	95	95	58	100*	71	67	58	92+
CHANGE-GET-MORE	Missing start	81	90	67	83	48	81*	58	83
	Missing change	57	100*	58	83+	33	86*	42	83*
CHANGE-GET-LESS	Missing start	52	62	58	42	52	57	42	42
	Missing change	95	100	67	83	76	81	67	83
	Missing end	86	86	67	83	62	71	58	67
COMPARE	Missing difference	100	100	75	75	67	67	67	58
	Missing small	86	86	67	92+	62	67	58	75
	Missing big	52	76*	50	42	43	57	42	42

Notes. * improvement significant at the .05 level.
+ improvement significant at the .10 level.

For Class B, correct strategy use increased for all subtypes except those that involved addition as the correct strategy (Change-Get-Less-missing- start and Compare-missing-big). Much of this across-subtype increase (53% of all improvement) was found to be due to the fact that several of the children subtracted for all problems on the posttest. However, the significant improvement in strategy usage found for the starred items in Table 1 were not due just to these children. Thus across both classes the teaching seemed to have affected the solutions of additive problem types (Put-Together and Change-Get-More) more than those of the subtractive problem types (Change-Get-Less and Compare).

Class A worksheet data were also analyzed for these variables. For the measure of correct solution strategy, a large effect of major problem type was observed, such that the ordering of difficulty from easiest to hardest was Put-Together, Change-Get-More, Change-Get-Less, and Compare, $\underline{F}(3, 72) = 5.05$, $\underline{p}<.01$. This parallels what is generally found for small (1-digit) problems. No significant differences were found, however, in the correct answer data, though the trends were in the same direction as the strategy data. The children from Class B who had subtracted on all problems on the posttest did not do so on worksheets. Thus these children did not simply learn always to subtract for word problems.

An important question that relates to the central focus of the study concerns the nature of the children's use of the pictorial representations in solving the problems. Children could, on the whole reliably select the correct representations, but this does not indicate that they actually used the representations as a solution aid. Although this is a difficult issue to address in the absence of detailed interview data, further analyses of posttest and worksheet data did reveal that there is a strong relationship between quality of the representation and correctness of solution strategy. Few correct strategies were observed in the absence of correct pictures, and few correct pictures without correct strategies. Further, except for Compare problems, few pictures were recorded that seem "trivial" (the two given numbers in the story simply filled into the first two available boxes, independent of story structure).

DISCUSSION

The data indicate that children did improve in their abilities to represent correctly and to solve certain addition and subtraction verbal problems. Children learned to use a subtractive solution strategy for the relevant problems with an additive semantic structure but did not succeed at those problem subtypes having a subtractive structure but requiring addition as the solution strategy. Some of the children manifested "subtract-only" behavior on the posttest, but they did not so behave on the worksheets. It seems possible that, under test conditions, the context of many other problems that required subtraction operations prompted the operation of the subtraction algorithm even though the current problem was understood as one requiring an answer greater than either of the two givens. In the present study, both demonstration and worksheet problems favored subtraction by a factor of two to one, simply because there exist twice as many kinds of problems requiring subtraction than requiring addition as the solution strategy. This ratio may bias responding toward subtraction.

Finally, although the gains obtained in problem-solving performance were not large, some impressive amounts of learning seem to have occurred. Especially in the high-quality class, children were by the posttest extremely proficient at identifying the class of word problem represented. At the beginning of instruction, when asked "What kind of problem is this?", children had classified problems simply as addition or subtraction problems. This classification permits confusion between the semantic structure of a problem as additive or subtractive and the required arithmetic solution procedure (addition or subtraction). The picture representations allow distinctions to be made between these aspects and provide an organization of the problem elements which may facilitate the decision concerning the arithmetic solution procedure. Thus the pictorial representations may provide a more general and flexible basis for the learning of the relationships between concrete situations and the arithmetic operations which describe those situations.

Carpenter, T. P., & Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh and M. Landau (Eds.), Acquisition of mathematical concepts and processes (pp. 7-44). New York: Academic Press.

Fuson, K. C. (1985). Teaching multi-digit addition and subtraction. In Proceedings of the Ninth Annual Meeting of the International Group for the Psychology of Mathematics Education. Noordwijkerhout, The Netherlands.

Fuson, K. C. (in press). Teaching subtraction as counting up. Journal for Research in Mathematics Education.

This research was funded by a grant from the Amoco Foundation.

SYMPOSIUM

SYMPOSIUM: Research Framework for Concept and Principle Learning - Revisited

Organizer: John Kolb, North Carolina State University
Presenters: John Kolb, North Carolina State University
Larry Sowder, Northern Illinois University
Patricia Wilson, Ohio State University
Lee Stiff, North Carolina State University

Within the past few years the number of research studies and research studies and researchers dealing specifically with concept and principle learning in mathematics appears to have diminished. This symposium is designed to renew the focus upon concept learning and principle learning research. Some new perspectives and approaches to this important research area will be offered. The goal of the symposium is the development of a framework in which future research efforts can be centered and the establishment of personal communications among researchers interested in the area.

This symposium will be a two hour session is which those attending will be given an opportunity to become actively involved in the discussion of ideas. Each presenter will serve as a catalyst for the exchanging of observations and ideas between the symposium leaders and participants. An informal follow-up session to the symposium will be scheduled for those participants who wish to continue the interchange.

DISTINGUISHING CONCEPTS AND PRINCIPLES IN MATHEMATICAL TEXT

John R. Kolb

North Carolina State University

Concepts, principles and identities are defined as types of mathematical content and their properties are discussed in the context of mathematics text.

Concepts and principles form the bulk of cognitive knowledge and the components of further learning. All relationships in mathematics are based in principles and principles in turn are formed from concepts. Mathematical symbols and words are signs with no meaning of themselves. They stand for something called their referents. The referents are the "real world" objects, events, actions, or constructions and these are represented with mathematical symbols and labeled with words in the natural language.

A *concept* is a set containing all referents which possess a combination of attributes commonly shared. Thus a concept is a class of things that can be responded to as one entity when the focus is on the characteristics that are common to all the members. To refer to the concept it is usual to associate with it one or more word labels or other symbols. When a referent or its representation from the set comprising a concept is displayed it is called an *example* of the concept. Any referent that does not belong to the set is called a *non-example* of the concept.

When a propositional statement is provided that identifies all the relevant attributes that are common to the set of referents then that verbal statement is called a definition of the concept. In summary, a concept is a class of referents, it has an associated label to designate it, any member of the set is an example, and a statement that identifies all the salient characteristics for membership in the set is a definition.

In Figure 1 at the end of this paper is a two page layout of a textbook page of mathematics with annotations in the dotted boxes. Three concepts are introduced in this material. The presence of concepts in text is signaled by their labels: central angle, intercepted arc, and inscribed angle. One example of each concept is provided and two nonexamples of inscribed angle is shown. Definitions are provided for two of the concepts.

A principle is some relationship between classes of referents and thus a relationship between concepts. In Figure 1, an example of a principle is "The measure of an inscribed angle is one-half the measure of the arc it intercepts." The concept of inscribed angle is related to the concept of intercepted arc. The concepts are connected to the tranformation of halving their measures. This suggests a more formal definition of a principle: A principle is an ordered relation consisting of a domain set of concepts, an operation or rule, and a range set of concepts. (Merrill and Wood, 1974, p.22).

A principle may be presented in mathematical text in either of two modes. A propositional statement can be given that names the domain concepts by their labels, and states the rule that relates them. In Figure 1, three principles are expressed as propositions (generalizations). In the second mode, a specific case of the principle can be shown by displaying a referent or representation from each of the concept classes in the domain and the range as well as the specific application of the relational rule. A specific case of a principle is called an instance of the principle. In Figure 1, an instance is given of the principle whose propositional form is "A central angle and its intercepted arc have the same measure in degrees." Sometimes principles will have name labels associated with them just like concepts. They have labels such as the Fundamental Theorem of Arithmetic or Divisibility Rule for Nine. In summary, a principle is like a function on the set of concepts. It may have an associated label, a specific case utilizing particular members of the domain and range concepts is called an instance of the principle, and a verbal statement that generalizes the rule in terms of the labels of the domain and range concepts is a propositional statement or rule generality. (Merrill and Wood, 1974, p.37)

A third type of content is a verbal association (Gagne, 1985, p.5) or identity (Merrill and Wood, 1974, p.19). An identity is a pair of entities that are associated one-to-one. The entities can be pairs like symbol to symbol, label to object, event to symbol, etc. Some identities in Figure 1 are the connection of the label "arc AB" and the symbol "AB" and the association of the letter "O" with the word phrase "center of the circle." To learn an identity is to be able to recall one member of the pair when presented with the other member as stimulus.

Concepts and principles may a) be formed by examples (instances) and non-examples (non-instances) that are encountered or b) be acquired by detecting the meaning conveyed by the concept's definition or the principle's propositional statement. These two methods of presentation appear to engage the learner in two different types of learning processes. The first mode which Sowder (1980, p.253) calls attainment requires the student to identify the salient attributes and use these to generate a classification rule for concepts or the transformation rule for principles. This is a kind of generative learning in which the student must devise an abstraction from a set of specifics. The second mode which Sowder (1980, p.254) terms assimilation requires the student to read meaning into verbal statements in an effort to make an abstraction more concrete and specific. It is an interpretative type of learning.

Gagne, R. M. (1985). The conditions of learning (3rd. ed.). New York: Holt, Rinehart and Winston, Inc.

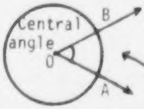
Merrill, M. D. & Wood, N. D. (1974). Instructional strategies: A preliminary taxonomy. Columbus, Ohio: ERIC/SMEAC.

Sowder, L. L. (1980). Concept and principle learning. In Shumway, R. J. (Ed.), Research in Mathematics Education pp.224-285). Reston Va.: National Council of Teachers of Mathematics.

IDENTIFYING TYPES OF CONTENT IN INSTRUCTIONAL MATERIAL

Angles in a circle.

A circle has many special angles associated with it. One important kind of angle is a central angle. A central angle in a circle is one that has its vertex at the center of a circle. In the circle below, ∠AOB is a central angle.



CONCEPT: Defined by a verbal statement. An example of the concept is given.

Recall that we always denote the center of a circle by O. Thus, the symbol for a central angle must always contain an O for its vertex.

IDENTITY: O and center of circle are interchangeable symbols.

IDENTITY: O and vertex of central angle are interchangeable.

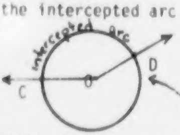
Each side of a central angle cuts the circle in a point, and these points form two arcs, one smaller than the other. (Remember that when we write $\widehat{AB}$ for arc AB, we mean the smaller of the two arcs formed by A and B.)

PRINCIPLE: A statement that gives a relationship between central angles and arcs of a circle.

IDENTITY: Arc AB same as $\widehat{AB}$

IDENTITY: $\widehat{AB}$ means smaller arc formed by A and B

The intercepted arc of an angle is the smaller of the two arcs it forms. In the circle below, $\widehat{CD}$ is the intercepted arc of the angle ∠COD.

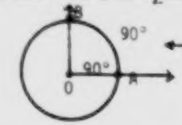


CONCEPT: Defined by a verbal statement. An example of the concept is given.

An important fact to remember is:

A central angle and its intercepted arc have the same measure in degrees.

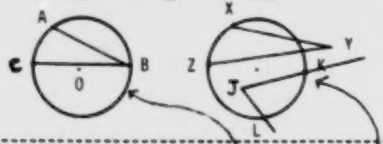
In the circle $m\angle AOB = 90°$ and $m\overset{\frown}{AB} = 90°$, so $m\angle AOB = m\overset{\frown}{AB}$.



PRINCIPLE: The rule or operation is equality. An instance is given of the principle.

There is an implied technique or prescription (some may call it a skill) that is not stated. Stated in a to do language, it is: To find mAB (where it is smaller than a semi-circle) construct its central angle, measure it with a protractor, and that will be the $m\overset{\frown}{AB}$.

$\angle ABC$ in the circle at the left below is called an inscribed angle. In the circle at the right, $\angle XYZ$ and $\angle KJL$ are not inscribed angles.



CONCEPT: This concept is not defined by a statement characterizing its attributes. Instead, one example and two non-examples are shown; and from this, the students are to learn the concept.

An important relationship is:

The measure of an inscribed angle is one-half the measure of the arc it intercepts.

PRINCIPLE: A statement that gives a relationship between inscribed angles and their intercepted arc.

Figure 1. Analyzing the types of content in a textbook-like presentation of a mathematical topic.

PRINCIPLE LEARNING - REVISITED

Larry Sowder
Northern Illinois University

Principle learning unfortunately has not received much attention. Some areas that invite exploration are given here, organized around the initial, basic learning of the principle, its incorporation into a memory network, and the improvement of its accessibility and retention in the network.

Interview studies to ascertain how students are thinking quite often have resulted in the dismaying realization that many students have at best a superficial grasp of their school mathematics. Indeed, Davis (1984) refers to such studies as "disaster" studies. Principles (relationships among concepts) are the bases for understanding algorithms and play central roles in solving many problems. It is clear that instruction in concepts and principles, the underpinnings for non-rote behavior, is falling far short of success.

Compared to concepts, principles have received much less research attention (cf. Shumway, 1980), and there does not seem to have been much research on principle learning as such in recent years. Hence, the following is a potpourri from my outlook, rather than a review of research in the area.

THE BASIC LEARNING OF A PRINCIPLE

Kolb's model for concept learning (Shumway, 1980, pp. 269-274) almost begs that a similar analysis be done for principle learning. No one seems to have carried out such an analysis, perhaps because there are more types of moves for principles (stating, justifying, instantiating, applying, for example - see Cooney, Davis, and Henderson, 1975) and it is arguable what criterion is most appropriate for principle learning.

The learning of concepts through examples and counterexamples has its counterpart with principles. Indeed, "discovery learning" is perhaps more appropriate for principles than for concepts in that verification of results in discovering a concept depends on someone other than the learner, whereas learners can (at least in theory) check their hypotheses about a principle. From examples alone, students could never be certain that their notion of the concept "trapezoid" is the conventional one. On the other hand, generation and checking of other cases can lead to a great confidence in, for example, the principle about equality of measures of alternate interior angles with parallel lines. Hence, it is disappointing that discovery learning, with principle learning as the object, is no longer fashionable.

Discovery learning can be viewed as a subset of problem solving. What can be easily lost under that view is attention to the claim that discovery learning results in greater retention. Do discovered principles fit into one's personal mental framework better than "told" principles do, and thus give greater accessibility during memory searches? Such a question is unlikely to be asked if problem solving, rather than principle learning, is the focus.

Is there a counterpart, for principles, to the notion of a rational set of examples and counterexamples for concepts? That is, can one design a set of instances and noninstances for a given principle which should equip the learner to apply the principle to, and only to, appropriate places?

FITTING A PRINCIPLE INTO A MEMORY NETWORK

Most of us would hope that, eventually, a principle could be related to other information and to new applications and problems. Most mental models recognize that many pieces of information can be linked in one's mind, and are linked by better-performing learners. Whether one calls such a network a semantic network, a schema, a frame, a template, or a script, the disaster studies make obvious that we need greater curricular and research attention to these networks. Either these networks do not exist for many learners, or the learners are not calling on them.

Are there "linking" moves teachers can use? What teacher actions, for example, might lead a student to relate the principle to a great number of other concepts, principles, applications, and problems?

Two aspects of the use of language seem worth examining further. (a) Would greater use of small groups give richer networks in learners? Rather than the learners relying only on the teachers' language, would the give-and-take possible in a small group (and at the learner's level of language), result in a less rote-driven grasp of principles? (b) Is it helpful to have a label for a principle, perhaps mnemonic in nature? School concepts have verbal labels, of course, as do some principles, such as the Pythagorean and binomial theorems, the "laws" from trigonometry, or properties like commutativity of multiplication. Occasionally one sees this idea used more extensively, as in Coxford and Usiskin's BAIT label (for "Base angles of an isosceles triangle are congruent") or their "side-splitting" label for a theorem about a line parallel to one side of a triangle (1971). Do such labels help, for example, in searching a network for a key to some problem, perhaps by giving a more "compact" representation of the principle? This question overlaps into the next section.

REMEMBERING A PRINCIPLE, AND MAKING IT ACCESSIBLE

Accessing concepts and principles already learned is often a key to solving problems. (Perhaps it was extensive practice at accessing such information that lay behind whatever success can be claimed for our pre-heuristics approach to the teaching of problem solving: Do lots of problems.) Thus the question of whether giving labels to principles would improve their accessibility is of great interest. Are there other methods that might enhance the accessibility of learned principles? Instruction at the time of the principle learning might be planned with later retrieval in mind. For example, a teacher might merely state, "Whenever you've got a problem with a triangle and angles in it, you should remember that the angles of the triangle add up to 180."

Periodic review is usually accepted as necessary for remembering information over the long term. Saxon's unfortunate advertising campaign may have obscured some promising information about the frequency of review that many be necessary to maximize retention (see, e.g., Klingele and Reed, 1984, but contrast Swafford, 1984).

Finally (and applying to all aspects of principle learning) is the establishment of a learner's intent to use, to remember, and to relate the principle. The finding that the purpose of a lesson for some young learners is simply to "get it done" is, lamentably, no doubt applicable to many older learners as well ("Do students learn from seatwork?" 1982). With so much instruction in mathematics centered on mastery of algorithms and generation of answers, it is no wonder that many learners expect to rely on, and to need only, their memories of "the way to do this kind" for all work in mathematics. Learner expectations must be re-formed through a redirection of the curriculum to higher-level thinking, perhaps through a genuine commitment to a problem-solving emphasis.

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CONCEPT AND PRINCIPLE LEARNING IN MATHEMATICS - REVISITED

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Researchers seek superior instructional strategies for teaching mathematical concepts and principles. A brief discussion of teaching strategies research with emphasis on the Kolb model of concept learning is presented here.

Henderson (1967) presented a taxonomy of teaching behaviors for teaching math concepts based upon classroom observations. Instructional dialogue between teacher and students were analyzed into identifiable segments called "moves". Teaching behaviors such as giving examples, providing definitions and comparing characteristics are instances of moves. There are E moves (giving examples or nonexamples) and C moves (stating analogies or definitions).

Henderson's taxonomy provides a means for analyzing and defining constituent moves of instructional dialogues. A sequence of moves used to teach a concept is a teaching strategy. The type, sequence and number of moves in a strategy characterizes the strategy. A sequence made up entirely of E moves is an E strategy. A sequence of E moves followed by C moves followed by E moves is an ECE strategy. Persumably, some strategies are better than others.

Researchers have tried to identify teaching strategies (Henderson, 1970) that are superior for teaching given mathematical concepts. However, no such instructional strategies have been identified (Cohen & Carpenter, 1980; Dossey, 1976, 1980; Dunn, 1983; Klausmeier & Feldman, 1975). Several possibilities might explain why no superior strategies have been found. Among these are (a) no real differences exist among strategies, (b) the right combination of moves has yet to be produced and (c) previous investigations have not been sensitive to actual differences among strategies. In any case, the need for a systematic approach to the problem of selecting teaching strategies existed (Kolb, 1977; Tennyson, Chao & Youngers, 1981).

One such approach was the Kolb model (1977) of concept learning. The model described the effects of strategies consisting of either all E moves or C moves and the relationship between these strategies while considering the learner's prior relevant knowledge of the concept being taught and the number of moves that made up the strategy. It was hypothesized that C moves would be used more effectively by learners with high relevant knowledge and that E moves would be more useful to learners with low relevant knowledge. In addition, it was reasoned that increasing the number of moves in an E strategy would increase the likelihood of learning the concept for learners with low relevant knowledge more than for learners with high relevant knowledge. In contrast, increasing the number of moves in a C strategy would not significantly affect concept attainment for learners at low or high relevant knowledge.

Gagne's (1970) type 6 and type 7 learning were used to define concept attainment. Type 6 learning is characterized by a learner's ability to sort instances of the concept, to produce new examples of the concept and to generate an informal working definition of the concept. Type 7 learning is characterized by a learner's ability to comprehend an idea or message conveyed by a verbal statement. Based upon the Kolb model it was hypothesized that an E strategy induces type 6 learning and a C strategy induces type 7 learning.

Several studies (Stiff, 1978; Weiger, 1978; Sikes, 1979) have examined the Kolb model of concept learning. In general, these studies support the hypotheses of the model subject to concerns about the methodology used in each study. Relevant knowledge was defined operationally in the studies in two ways: by exposure of contrived concepts subordinate to the contrived concept to be learned or by achievement on relevant knowledge tests administered to determine groupings of low, medium and high by which learners could be identified. Using either method to define relevant knowledge overlooks the learner's ability to process written information, to memorize detail not to mention the learner's I.Q. Number of moves as a factor in the studies is not well-defined in that it has not been demonstrated that all moves are equally effective in producing concept attainment. For instance, a strategy of two example moves and one non-example move may be more powerful than a strategy of two analogies and one single characteristic move even though each strategy consists of three moves.

Research in principle learning has received less attention than research in concept learning (see Shumway, 1980). Classifications of "moves" for principle learning (Cooney, Davis & Henderson, 1975) may be useful to future research efforts, particularly if the Kolb model of concept learning is significant in explaining how students learn math concepts. There may be a Kolbian model for principle learning that explains how students learn the relationships among concepts that form mathematical principles.

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RESEARCH METHODOLOGIES FOR CONCEPT AND PRINCIPLE LEARNING

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Rationalistic and naturalistic methods of inquiry are compared and common criteria for rigor are discussed. Improving concept learning research through triangulation is suggested.

Concept learning and principle learning are extremely complex. Researchers in these areas have the task of finding, adapting or creating methodologies that will probe the learner's mind and extend the present knowledge about how concepts and principles are learned.

diSessa defines science as refined intuition. He explains that since science is integrated, it is not as context dependent as common sense. There is a depth and richness of intuitive knowledge that can be developed into scientific thinking. We need "... to convey the sense of incredible complexity, interrelation and depth of scientific knowledge as compared to commonsense reasoning" (diSessa, 1985, p. 17). Although diSessa was addressing the responsibilities of teaching science, his comments are relevent to research. We need to construct better methodologies that will explore concept and principle learning. The research techniques will necessarily be complex, interrelated and probe deeply; research should not be as context dependent as common sense.

RATIONALISTIC AND NATURALISTIC INQUIRY

Rationalistic inquiry is often conducted using quantitative methodology and naturalistic inquiry is linked to qualitative methods. The separation does not seem necessary since both methods can and should contribute to both rationalistic and naturalistic inquiry.

Guba (1981) claims that rationalistic and naturalistic inquiry do differ in the following areas.

Philosophy

The rationalistic paradigm assumes there is one reality and that inquiry can converge. For example, it is appropriate to single out one variable for study and to combine information in order to approach truth. The naturalistic researcher assumes multiple realities and that inquiry will diverge as more is learned. Believing that all variables are related, the study of one variable is senseless.

Quality Criterion

The rationalistic approach demands rigor and assures it by controlling or randomizing variables. The naturalistic approach seeks relevance and looks for external validity.

Source of Theory

Hypotheses and questions are generated and tested by researchers using a rationalistic paradigm. Theory emerges from the data collected in a naturalistic paradigm.

Knowledge Explored

Restricted by instruments and hypothesis testing, rationalistic researchers operate at the level of propositional knowledge. Naturalistic researchers are able to explore tacit knowledge such as intuitions, apprehensions, or feelings that can not be expressed in language.

Instruments

In order to obtain objectivity, rationalistic inquiry uses instrumentation that protects the subject from the influence of the researcher. Naturalistic researchers use themselves as instruments in order to gain flexibility and insight.

At first glance, these differences seem to make the two paradigms completely incompatible. Closer inspection offers some possibilities for collaboration. For example, naturalistic inquiry generates theories and rationalistic inquiry tests hypotheses. The fit seems natural, but we should not stop with this limited effort to combine two powerful ways of investigating learning.

CRITERIA OF RIGOR

In order to draw knowledge from different paradigms, there must be a system of measuring accountability and a vocabulary that accomodates both systems of inquiry. Guba and Lincoln (1981) suggest that the following tests of rigor are appropriate for both rationalistic and naturalistic inquiry.

Truth Value

How do you establish confidence in the "truth" of the findings in the given context? Rationalistic inquiry uses measures of internal validity and naturalistic inquiry uses measures of credibility.

Applicability

How do you determine how well the findings apply in another context or with other subjects? Rationalistic inquiry seeks external validity and generalizability, and naturalistic inquiry focusses on transferability.

Consistency

How do you determine if the findings would be consistent if the inquiry were replicated? Realiability is an important criterion for rationalistic researchers. Naturalistic researchers believe that instability is natural making the task of measuring consistency difficult. They measure dependability which includes both stability and trackability of explainable changes.

Neutrality

How do you determine to what degree the findings are related to the subjects and context and to what degree the researcher influenced the inquiry? The rationalistic researcher strives for objectivity. The naturalistic researcher looks for confirmability which shifts the focus from the investigator to the data.

TRIANGULATION

Triangulation is the combination of methods in order to study the same phenomenon. The term came from the navigation practice of using geometry and multiple viewpoints to improve the accuracy of a course. By employing different paradigms and different methods, we can improve the accuracy of research in concept and principle learning. "Triangulation may be used not only to to examine the same phenomenon from multiple perspectives but also to enrich our understanding by allowing for new or deeper dimensions to emerge" (Jick, 1983, p. 138)

In the theory, triangulation offers the possibility of interfacing much of the current rationalistic research in concept acquisition with the more naturalistic research in how knowledge is constructed by various individuals. There are problems to overcome. The theory assumes that the weaknesses of a particular method will be compensated for by another method. We must be certain that combining methods increases the assets rather than compounding the liabilities. We must also guard against artifically using one paradigm as an add-on to strengthen weak findings from a poorly conceived study.

Pragmatically, we must consider how to implement triangulation. At the present time there is limited information on how to proceed. Most researchers have been trained in one method and are fairly naive about other options. Journals tend to specialize in one methodology and discourage mixed-breed research. Since vocabulary differs between paradigms, even informal communication is difficult.

Jick (1983) offers some reasons for pursuing the possibilities of triangulation. If the different methods of research present convergent findings, the researcher has greater confidence in the truth value of the research. If the combined research presents divergence results, the researcher is challenged to alternative and more complex explanations. Diverse theories may spawn innovative methods of inquiry. By integrating multiple methods, researchers become more sensitive to all the issues of a study.

IMPLICATIONS

In concept and principle learning research, we can begin to address the pragmatic needs of using multiple methods by explicitly defining terms we use as well as methods of investigation. Research programs can be networked so that a variety of research teams approach the same problem with different methodologies. Individual research projects should include qualitative and quantitative phases. Research reports need to discuss fully the limitations of studies. As we create new methods of inquiry and data analysis, we must maintain high standards of rigor that are mutually respected.

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SYMPOSIUM: THE MULTIFACETED COGNITIVE DOMAIN OF ADOLESCENTS AND ADULTS: IMPLICATIONS FOR TEACHING MATHEMATICS

Organizer: Dorothy Buerk, Ithaca College
Presenters: Dorothy Buerk, Ithaca College
Roberta Dees, Purdue University Calumet
Margaret Farrell, State University of New York at Albany
Janet McDonald, State University of New York at Albany
Discussant: Joan Mundy, University of New Hampshire

The symposium will focus on the cognitive domain of adults and adolescents and on the issues that are unique to this age range and that are raised by our diverse work. Once people develop the abilities to think abstractly and come in contact with the complexities of the world of mathematical ideas, the research questions and research issues change dramatically from those posed for younger learners. We will look at particular reasoning abilities of adolescents and young adults, students' views of mathematics as a field of knowledge, the way students apparently structure their knowledge, and the impact of cooperation on learning. We are concerned with what students bring to the learning situation in each of these areas and have both quantitative and qualitative data to present. We have chosen diverse theoretical frameworks to look at different facets of the cognitive domain. We are particularly concerned with the implications for teaching and learning and for curriculum modification that could result from this research.

ADULT CONCEPTIONS OF MATHEMATICAL KNOWLEDGE*

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My research with adult women and college students of both genders, most of whom would prefer not to be in mathematics classes, has shown that for many of these people mathematics is a collection of right answers with correct methods and exact symbols. While this may be secure for those who can correctly use the symbols, it is devastating for those who cannot. For them the struggle is to memorize symbols and processes that have no meaning. As the symbol systems and processes become more complex, they become more difficult to memorize. For these people, mathematics becomes sheer magic, a magic of which they are in awe, but a magic which they cannot perform.

A woman colleague, not in mathematics, expresses this view of mathematics in a creative and delightful way:

> On the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it into a fence, forty cubits high, and infinite cubits long. And on this fence, in fair capitals, he did print rules, theorems, axioms and pointed reminders. "Invert and multiply." "The square on the hypotenuse is three decibles louder than one hand clapping." "Always do what's in the parentheses first." And when he was finished, he said "On one side of this fence will reside those who are good at math. And on the other will remain those who are bad at math, and woe unto them, for they shall weep and gnash their teeth."
>
> Math does make me think of a stainless steel wall--hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put your nose against it, it doesn't take your shape, it doesn't have any smell,

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*This work was supported by the Mina Shaughnessy Scholars Fund of the Fund for the Improvement of Postsecondary Education.

> all it does is make your nose cold. I like the shine of it--it does look smart, in an icy way. But I resent its cold impenetrability, its supercilious glare. (In Buerk, 1982)

Her words serve as powerful metaphor for me and indicate clearly two very distinct issues which are problematic for many as they experience mathematics. One issue is the view that mathematical knowledge is absolute and all known. The second concerns the desire to find a way to make a connection with the material in some personal way--to gain a "handhold."

MATHEMATICS AS ABSOLUTE KNOWLEDGE

Many people I meet, in and out of my classroom, believe that mathematics is made up only of rules, formulas, and proofs to be memorized; skills to be practiced; and methods to be followed precisely. They believe that mathematics is a discipline where certainty is secure; where all questions have answers which are known to authority (mathematician, professor, TA, textbook); where memorization, hard work, and some mystical quality called a mathematical mind are required.

This conception of knowledge is called "dualistic" by psychologist William G. Perry, Jr. (1970, 1981). His theory of intellectual growth suggests a sequence of ways that college students and adults view knowledge. His theory gives us a frame of reference to use in interpreting this conception of mathematics as a field of knowledge. He defines dualism as:

> Division of meaning into two realms--Good versus Bad, Right versus Wrong, We versus They, All that is not Success is Failure, and the like. Right Answers exist somewhere for every problem, and the authorities know them. Right Answers are to be memorized by hard work. Knowledge is quantitative. Agency is experienced as "out there" in Authority, test scores, the Right Job. (1981, p. 79)

This dualistic view of mathematical knowledge was made very clear to me recently. In a class discussion on exponents, an eighteen-year-old freshman told me that exponents were added when multiplying factors with the same base. I asked him way. He said, "That's the rule." I asked him why the rule said that. "It just does," he replied, "it's the rule I was taught." But "why?" I asked again. He looked at me very seriously and asked, "You mean there's a reason?"

A woman whom I have interviewed indicates that her dualistic view was reinforced by her teachers.

> I think of math problems or situations as having right and wrong answers (very black and white), but having a variety of ways to reach the answer. Unfortunately, my math teachers never stressed the fact there could be more than one way to approach a problem. For this reason, and there are other reasons, I do not see math as a "creative activity." It is most definitely not linked to language, or music, or the other humanities. (In Buerk, 1981)

MATHEMATICS AS AN INHUMAN CREATION

The colleague I quoted earlier does not recognize the "person-made" quality of mathematics, but views the knowledge as handed down as if by God, as a finished product--a view that prevents her from finding a way to relate to it. She finds, therefore, that mathematics "offers no handhold," "it doesn't take your shape." Even those of my colleagues and students who do acknowledge that mathematicians create mathematics believe that the proofs that verify mathematical statements come out of the heads of mathematicians full-blown--like Athena from the head of Zeus. They believe that the succinct, formal statements which clarify mathematical ideas represent the way the minds of mathematicians work. (See Buerk, 1985a.)

STRATEGIES TO ENHANCE MEANING MAKING

In my twenty-five years of teaching in various settings (two-year college, four-year private college, state university division of continuing education, experimental high school, overseas military base, maximum security correctional facility, and individual tutorials with women) I have developed strategies to help students to move away from a dualistic view of mathematical knowledge, to become aware of the person-made quality of mathematics, and to develop confidence in their own ability to do mathematics. These strategies include: placing topics in their historical context, acknowledging and encouraging alternative methods and approaches, encouraging collaboration in mathematics learning, making concerted attempts to avoid absolute language, offering opportunities for students to reflect on paper about their ideas and feelings about mathematics. (See Buerk 1985b for a complete listing of these strategies.)

These strategies are the basis for intervention programs I developed to change the conception of mathematical knowledge in adolescents and adults. With the change in conception of knowledge come changes in confidence and a reduction in mathematics avoidance. Individual experiences from a study of adult women (Buerk 1981, 1982), a writing seminar in mathematics, and a basic skills course will be presented.

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HOW DOES WORKING CO-OPERATIVELY HELP STUDENTS IN INCREASING PROBLEM-SOLVING ABILITY?

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This paper describes a clinical study, or teaching experiment, conducted during spring semester, 1985, in a pre-algebra developmental mathematics course to explore how college students work together to solve word problems. While co-operative learning may yield gains in students' higher cognitive thinking, the attitudes of students may inhibit true co-operation. In this study we attempted to investigate and address these attitudes and to develop a recordkeeping system to simplify the management of a co-operative classroom.

There are several definitions of co-operative learning. In one approach (Slavin, 1980), students work in teams to master mathematics content, sometimes competing with other similar groups. Within groups, students work individually on their own learning goals, assisting each other by checking work, drilling each other and tutoring when possible. In the other major approach (Johnson and Johnson, 1975), students work together on the same learning goal and produce one end product or solution. In this method, students perceive that they can attain their goals if and only if other team members also attain theirs. This method, using co-operation as a mode of learning, is the intended one in this study.

Some studies seem to indicate that the greatest benefit of the co-operative method may accrue in complex tasks, such as concept learning and problem solving (Cohen, 1982; Dees, 1983; Sharan and Hertz-Lazarowitz, 1980; Webb, 1978).

In discussing the nature of learning in small groups, Webb (1982) reviews four studies on student interaction and achievement in mathematics. Students were in seventh, eighth, ninth and eleventh grades. Giving and receiving help were categorized as either explanations or terminal responses (giving the correct answer with no explanation or pointing out an error with no explanation of how to correct it). Webb found that giving explanations was beneficial for achievement but giving terminal responses was not. Furthermore, receiving explanations tended to be positively related to achievement but receiving terminal responses and receiving no response to a request for help were detrimental for achievement. Webb observes that the composite variables "giving help" and "receiving help" may not be meaningful, since the positive and negative components of each may cancel each other.

The questions raised by the studies include the following: How does working co-operatively help students in increasing their skill in problem-solving? Does it matter which students work together? How can an instructor observe and record the way in which students work together?

In a teaching experiment, the researcher works intensively with a small number of students. Attempts are made to modify and update the procedures as the experiment yields information.

THE TEACHING EXPERIMENT

Students were 14 adults in an intact developmental mathematics class at Purdue University Calumet. The course consists of two 8-week segments, Arithmetic Skills and Pre-Algebra Problem-Solving Skills. Students were given a battery of diagnostic instruments at the beginning of the semester. The class meets 6 hours per week; about half is lecture-discussion and the rest is laboratory, in which students are working individually, with partners, or in small groups, with the help of the instructor and/or teaching assistant.

Data was kept about how many times students worked together, with whom, and who had which role (Helper, Helpee or relative equals). Efforts were made to see that everyone had an opportunity to work with everyone else in the class and to play different roles, sometimes helping and sometimes being helped. Shortly after midterm, students were interviewed individually concerning their progress and how they viewed working together, especially in solving word problems. The interviews typically took a half-hour or more and were audio-recorded.

RESULTS AND FINDINGS

We developed a chart (Figure 1) which is fairly easy to maintain and which holds information about who has worked with whom, how often, etc. The instructor can enter the initials of a student's partners or group members. We entered a "+" when we thought a person's role was primarily that of a Helper, "-" when the person seemed to be basically a Helpee, and nothing if we weren't sure or they seemed to be equal. The "working together" record should be on a separate sheet from the grade book because of the space required; we used pages from large bookkeeping ledgers. After a few weeks we could tell at a glance whether someone had always been absent when we had co-operative activities or with whom students had worked.

	1/21	1/23	1/28	2/2	2/5	• • •
Carol Anderson	1		CD	$\overset{-}{\text{D}}\text{H}$	$\text{J}\overset{-}{\text{M}}$	
Lisa Andretti	2		YW	$\overset{-}{\text{M}}\text{R}$		
Chris Doolittle	2^+	$\overset{-}{\text{J}}\text{M}$	CA	$\overset{+}{\text{TJ}}$	AT	
Bennie Griffith			MR	$\text{J}\overset{+}{\text{M}}$	EH	
⋮	Group of 4	Free	Assigned	Assigned	Free	

Figure 1

From observations and interviews, formal and informal, we received some insights.

I. Most of the students in developmental classes can be placed in one of the following two categories:

A. Highly motivated, often mature students. They are high in drive and ambition, but may be low in skills and confidence.

B. Unmotivated, directionless, often just out of high school or unemployed. Sometimes they are being sent to school by parents or others.

When students were required to work together, certain generalizations can be made about their reactions. Group A students were independent persons, determined to master the material. They did not mind helping others, once they felt confident that they understood it themselves and as long as the person to be helped wanted their help. They usually were not eager to co-operate with someone that they felt was not serious about the task at hand. Group B students, though sometimes lazy and apathetic, usually enjoyed the interaction and attention they received in the group (or from a partner); they usually responded to being helped by trying to hold their ends up. They sometimes took a passive role, merely copying the leader's solutions. They did not initiate much, but often assumed the role of cheerleader, encouraging or complimenting other students who made suggestions.

II. The students helped me arrive at models (shown in Figure 2) to illustrate what they had been doing and what I was asking them to do. Model I shows working in a co-operative manner, while Model II shows the results of actually working together, or us ing co-operation as a learning mode. We observed that in this class, Model II was used only when I required it. (This was accomplished by giving teams one answer sheet among them so that they were forced to collaborate; simply asking them to work

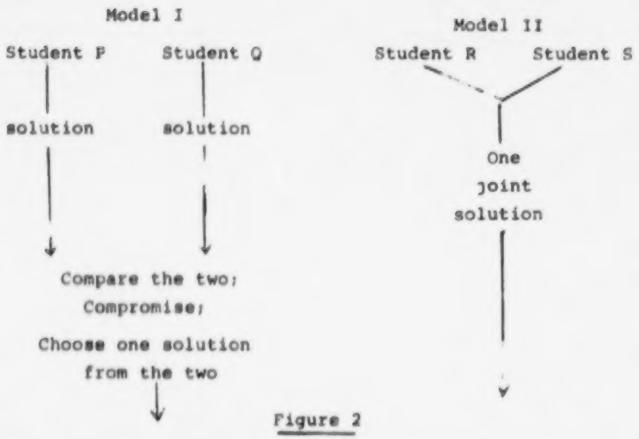
Model I
Student P
Student Q
solution
solution
Compare the two;
Compromise;
Choose one solution
from the two
Model II
Student R
Student S
One
joint
solution

Figure 2

together tended to produce only Model I behavior.) During the interviews we discuss the reasons for this; students often cited their previous training to "keep your eyes on your own paper," especially in mathematics classes.

III. With regard to questions raised earlier, I make the following conjectures:

a. Working co-operatively forces students to actually attend to the problem at hand. Peer pressure to help is motivational.

b. Discussing what the problem means clarifies it for the speaker as well as the listener.

c. Women in remedial or developmental matheamtics courses are often lacking in confidence; working together seems to increase their confidence. Peers in the class were lavish with praise when someone did a good job. Also, the women seemed to either not mind or actually relish the helping role. Thus, in trying to help others, they were also helping themselves.

d. Students generally do not know how to work together; they need instruction in this. Also, by sometimes choosing who works with whom, the instructor can assure that students sometimes have the role of Helper, sometimes of Helpee.

IV. No harmful effects of the co-operative method were observed. In addition to the usual affective benefits expected, I believe that the co-operative method has great potential for helping students to learn difficult mathematics concepts.

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COGNITIVE DEVELOPMENT AMONG ADOLESCENTS AND YOUNG ADULTS

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The studies referred to in this paper were completed by doctoral students or faculty at the State University of New York Albany over the period from 1980 to 1985. Cognitive development of adolescents and young adults was one of several variables studied by these researchers who were investigating curricular or instructional issues in mathematics or science education. In most cases, the kind of reasoning demonstrated by the subjects in response to particular measures was the central issue for these researchers, rather than classfication of subjects by stage.

INSTRUMENTATION

Several group classification instruments were used across more than one study. These merit particular attention here. Mr. Tall and Mr. Short (Karplus, Lawson, Wollman, Appel, Bernoff, Howe, Rusch & Sullivan, 1977) measures first-order direct proportional reasoning. A second commonly used group instrument was Longeot's Test of Formal Reasoning (Longeot, 1962, 1964). From a mathematics teacher's point of view, Longeot's three subtests assess one of the following: proportional or probabilistic reasoning, hypothetico-deductive reasoning, or reasoning about permutations and combinations. The third commonly used group test was the Test of Logical Thinking, or the TOLT (Tobin and Capie, 1981). The TOLT consists of ten items, two each designed to assess one of five reasoning abilities: proportional reasoning, controlling variables, probabilistic reasoning, correlational reasoning and combinatorial reasoning.

Three of the fifteen Inhelder and Piaget (1958) tasks were used in more than one study. These were the Chemical Solutions, the Bending Rods and the Projection of Shadows tasks. They were designed to measure combinatorial reasoning, controlling variables and proportional reasoning, respectively. In Table 1, each of these common measures is listed with the kind of reasoning which it appears to assess. Decisions on the reasoning classifications were reached on the basis of achievement test results matched with performance on the measures and assessments of construct validity, when available, rather than on the author's title of the measure. For example, Longeot used the label "test of combinations" for a set of problems which include permutations and combinations. Further, even the combination items do not fully test combinatorial reasoning which is more complex than merely listing, in a patterned way, all the possible combinations in some finite set.

Table 1
Kind of Reasoning Assessed by Each Measure

Instrument	Reasoning
Tall-Short Puzzle	Assesses first order direct proportional reasoning (metric) and understanding of non-standard unit.
Volume Puzzle	Designed to assess effect of submerging different weight, but same volume, ball in water. May assess recall of relationship from science class.
Mealworm Puzzle	Requires understanding of controlling variables and ability to analyze design.
Longeot-Logic	Assesses ability to reach an appropriate conclusion given premises in story form.
Longeot-Proportions	Requires ability to choose the more likely of two situations--each described in terms of two factors.
Longeot-Combinations	Assesses ability to count or list different orders (permutations) or different collections (combinations)
Chemical Solutions	Assesses ability to recognize all possible combinations and to use the results to decide next steps or to explain pairs of results.
Shadows	Assesses qualitative direct and inverse proportions and metric joint proportion in four successive subtasks.
Rods	Assesses understanding of controlling variables.
TOLT	Two items each assess proportional, probabilistic, correlational and combinational reasoning, and controlling variables.

There will be no attempt in this summary to include detailed information on the reliability or the validity (construct, predictive,...) of the cited measures. Nor will detailed information on test administration, interview training, administration and test scoring be included here. Such information is available in the original references.

RESULTS FROM STUDIES

A summary of the results of those studies which used one or more of the measures listed in Table 1 is included in Table 2. The students in each sample were classified in the Piagetian vernacular, as concrete operational (CO), transitional (TR) or formal operational (FO). In some cases, further data on student strategies and error patterns were analyzed and may be found in the original source. For the purposes of this report, the code (FO) found in Table 2 always indicates success on the type of reasoning assessed by the measure. The code (CO) always represents, at best, performance characterized by reliance on trial and error approaches, inductive reasoning and dependence on concrete experiences or familiar objects or events. The code (TR), as one would expect, represents behavior developing from early concrete operational to full or late formal operational. In some of the paper-pencil tasks, TR may include both late concrete operational and early formal operational behaviors. On the Projection of Shadows task, scoring allows for both early and late, concrete (ECO and LCO) and formal (EFO and LFO) operational.

In addition to the summary in Table 2, all researchers provided other data on students' performance on tests of mathematical reasoning and achievement. In his study of instructional modes at the college level, Pluta (1980) tested student achievement on unit emphasizing mathematical structure. Pluta found that for students classified as TR or FO, mathematical learning was enhanced by instruction which incorporated active manipulation of physical objects and an inductive approach while CO students were unable to achieve a satisfactory level of understanding, regardless of treatment. In a study of the validity of several paper-pencil measures, Farmer, Farrell, Clark & McDonald (1982) administered three original Piagetian tasks, the Chemical Solutions, Projection of Shadows and Bending Rods (Inhelder & Piaget, 1958). Only the data for the Shadows is displayed in Table 2. Results for all three tasks are given in Table 3.

Table 2

Percentages of Students Classified on Measures by Study

Study	Tall-Short	Volume	Mealworm	Longeot	TOLT	Shadows
Pre-Service El. Teachers Pluta (1980)				(N=48) CO=40 TR=31 FO=29		
9/10th Grade MSC Ss Farmer et. al. (1982)	(N=506) CO=65 FO=35	(N=506) CO=39 FO=61	(N=506) CO=87 FO=13	(N=506) CO=34 FO=66		(N=69) ECO=10 LCO=86 EFO= 1 LFO= 3
10th Grade Geometry Ss McDonald (1982)				(N=150) CO=19 TR= 0 FO=81	(N=136) CO=33 TR=30 FO=37	
Nigerian Form 3 Ss Fajemidagba (1983)				(N=99) CO=84 FO=16	(N=99) CO=84 TR=16 FO= 0	
Gifted Middle School Ss Farmer (1983)	(N=30) CO=13 FO=87			(N=30) CO= 0 FO=100	(N=30) CO=20 FO=80	(N=30) ECO= 0 LCO=74 EFO= 3 LFO=23
10th, 11th 12th MSC Ss Farrell and Farmer (1985)	(N=901) ECO=26 LCO=21 FO=53					(N=128) ECO= 4 LCO=72 EFO= 7 LFO=17

Table 3
Percentages of Students Classified (N = 69) on Three Inhelder and Piaget Tasks

Classification	Task Chemicals	 Shadows	 Rods	Total
Concrete	76.8	95.7	63.8	84.1
Formal	23.2	4.3	36.2	15.9

McDonald (1982) obtained estimates of students's cognitive structure with respect to the topic of similarity and compared these with the ways experts and their own teachers structured the similarity material.

Fajemidagba (1983), in a study of achievement on ratio and proportion problems, found that TR students succeeded on first order direct proportions when the items included concrete referents and real world examples. In a study of reasoning displayed by gifted middle school youngsters, Farmer (1983) gave data for each of the reasoning areas purported to be assessed by the TOLT (Table 4).

Table 4
Frequency of Success of Gifted Middle School Students (N = 30) by TOLT Reasoning Sections

Reasoning	Both Items	Exactly One	Neither Item
Proportional	22	4	4
Controlling Variables	23	1	6
Probabilistic	20	7	3
Correlational	15	12	3
Combinatorial	12	10	8

In order to follow up questions raised by performance on proportionality in several of these studies, Farrell and Farmer (1985) designed a study which focused on older students with more course experience in mathematics and science and evidence of success in the area of direct proportions. Students (N = 901) enrolled in tenth, eleventh or twelfth grade college-bound mathematics and/or science classes were administered Tall-Short. From the successful group of 474 students, a random subsample of 128 was interviewed on the Projection of Shadows task. (Table 2) The responses of the subsample were analyzed to identify the effects of feed-back and second trials and the effect of centering on direct proportions. It is especially revealing to note that only 7 of the 22 late formal students succeeded on the first trial for each subtask. Thus, feedback and the opportunity to try again were necessary so that these students could demonstrate their optimal competence.

ISSUES

An examination of the reasoning patterns listed in Table 1 shows that all of these are of concern to mathematics and science educators. A command of most of the reasoning strategies is necessary for meaningful learning of the typical mathematics curriculum required at the grade levels tested. What is particularly disturbing is the level of performance of college-bound "advanced" students. To what extent is the cognitive development of these students being retarded by the curriculum, instructional approaches or the usual classroom tests of learning? Are teachers' erroneous expectations of students' competence contributing to the failure of otherwise capable students to develop higher cognitive skills? These are some of the issues to be raised in the symposium.

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STABILITY OF COGNITIVE STRUCTURES AND RETENTION OF GEOMETRIC CONTENT RELATIVE TO COGNITIVE STAGE

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This study was designed to investigate the stability of cognitive structure of content by comparing cognitive "maps" generated by high school geometry students with those generated by the same students one year earlier and to determine the impact of structural differences on long-term retention of the subject matter. The results of the analysis indicated that the content structures created by subjects who were formal operational when they learned the material were significantly more stable than those who were concrete at the time of instruction. Those same subjects also retained significantly more geometry content.

INTRODUCTION

In a previous study, this researcher investigated the role of cognitive stage in the development of cognitive structures of geometric content (McDonald, 1982). That research indicated that cognitive "maps" of geometric content formed by formal operational subjects were significantly more like those of subject matter experts than the maps formed by concrete operational subjects. The purpose of the present study was to investigate the stability of those cognitive structures by analyzing the cognitive maps of the same subjects one year later, and to determine the impact of structural differences upon the long-term retention of subject matter content. If the significant differences between formal and concrete operators in the previous study were based upon a reliance of the concrete operators on "rote" memorization, then re-examination of the same subjects at a later date should yield significant differences in both the structural stability and subject matter retention as a function of cognitive stage.

METHODOLOGY

Subjects: Subjects consisted of the 40 subjects from the previous study drawn from a pool of 161 tenth grade "Regents" geometry students from a suburban New York State school. Twenty had been classified as concrete operational and twenty had been classified as formal operational.

Instruments: Classification of subjects was accomplished through the use of Sheehan's (1970) adaptation of the Longeot Test of Formal Operations (1964) and the Test of Logical Thinking (TOLT; Tobin & Capie, 1981). The instrument designed to map expert and student structure consisted of a similarity judgment task built from 13 terms chosen from the similarity unit in geometry. The subjects were directed to make pairwise similarity judgments for each of the 78 combinations of terms by assigning a number representing the degree of relationship. Patterns of cognitive structure were obtained through multidimensional scaling analyses of the similarity judgments. An expert target matrix, formed by consensual agreement among six mathematics educators, was used as a model of the subject-matter structure and as a basis of comparison with corresponding student structures of the same content. All subjects were also given a test on geometry content.

Procedure: Students were administered the similarity judgment task one year and one week after the original administration. Students were requested to make each of their judgments considering the relationship of the terms in the similarity unit. One week later, students were administered the content instrument. They were requested to make an attempt to complete each question even if they felt that they might have forgotten the material from the previous year.

DATA ANALYSIS AND RESULTS

Data Analysis: Each student's matrix of proximity judgments was standardized and distance coefficients were determined for each pair. The resulting distance matrices were subjected to nonmetric multidimentional scaling analysis using a four-dimensional MDSCALE solution. The new student maps were then compared to the original expert and student maps. The rank order correlations between each of the students and the target matrices were determined, converted to z-scores, and tested for significance using a one-tailed t-statistic.

The content unit test was scored by assigning partial credit to the numerical problems and to the proofs. Total raw scores and subtest scores were standardized and correlated with the target correlation scores from the MDSCALE analysis.

Results: The results of the analysis of the stability of structure indicated that the structures of the formal operational subjects were significantly more stable (t(38) = 5.25, p .001) and remained significantly closer in structure to subject matter experts (t(38) = 6.36, p .001) than those of the concrete subjects. The formal operational subjects also retained significantly more content than the concrete operational subjects (t(38) = 2.68, p .01). On the True/False subtest of the content test, the two groups showed no significant difference. Differences on the numerical problems were significant at the .025 level (t(38) = 2.56). Differences on the proof where students only filled in the reasons were significant at the .01 level (t(38) = 2.55) while differences in the proof where students supplied both statements and reasons were the most significant (t(38) = 3.66, p .0005).

DISCUSSION

Figure 1 shows a composite of three cognitive maps. The expert map was derived for the original study and used for this analysis as well. The vertical dimension represents a range of equality, the positive pole being most equal quantities and the negative pole being least equal quantities. The horizontal axis represents the whole versus part dimension with whole figures at the positive extreme and their parts at the negative extreme.

In comparison to the expert map, the prototypical concrete map from the initial study (not shown) was much more confused and compressed on the whole versus part dimension and indicated a general confusion of several significant terms. As shown in Figure 1, the prototypical concrete group member during the follow-up study, departs even more drastically from the expert map. Here most similarity in dimensionality to the expert map is lost.

The prototypical formal representations from both studies more closely approximate the expert target than either of the concrete maps. Although a certain amount of clustering of terms resulted in a compression of the axes for these subjects during the follow-up study (Figure 1), the extreme clustering apparent in the concrete maps is not as prominent.

The lack of significant differences in the True/False items is probably due to the guessing factor and their relatively simple content. That less statistically significant differences were found in the numerical problems than in the proofs is probably a result of their being algebraic in nature. The general inability of concrete students to be able to successfully complete the proofs is a factor correlated significantly to the lack of an integrated structure representing the interrelationships of the required concepts. The concrete subject was unable to relate terms outside of the given cluster to terms within the cluster, and it is these types of interrelationsips that are the basis of proof.

CONCLUSIONS AND IMPLICATIONS

The combined findings of these two studies would indicate that there are inherent qualities in the content of high school geometry which make it extremely difficult for certain students to develop meaningful cognitive structures of its concepts. As a result, it appears that these students may be forced to learn the material by rote methods, resulting in unintegrated cognitive structures and lack of retention. To promote the cognitive development needed, changes in instructional methods, modes, and strategies to match the cognitive developmental level of the student would seem mandatory. The results are also indicative of the importance of consideration of the reciprocal implications between knowledge representation and general control strategies as students develop an understanding of any abstract domain. Efforts to understand the acquisition of knowledge might benefit from application of a control systems framework or from the use of microcomputer simulation and graphics.

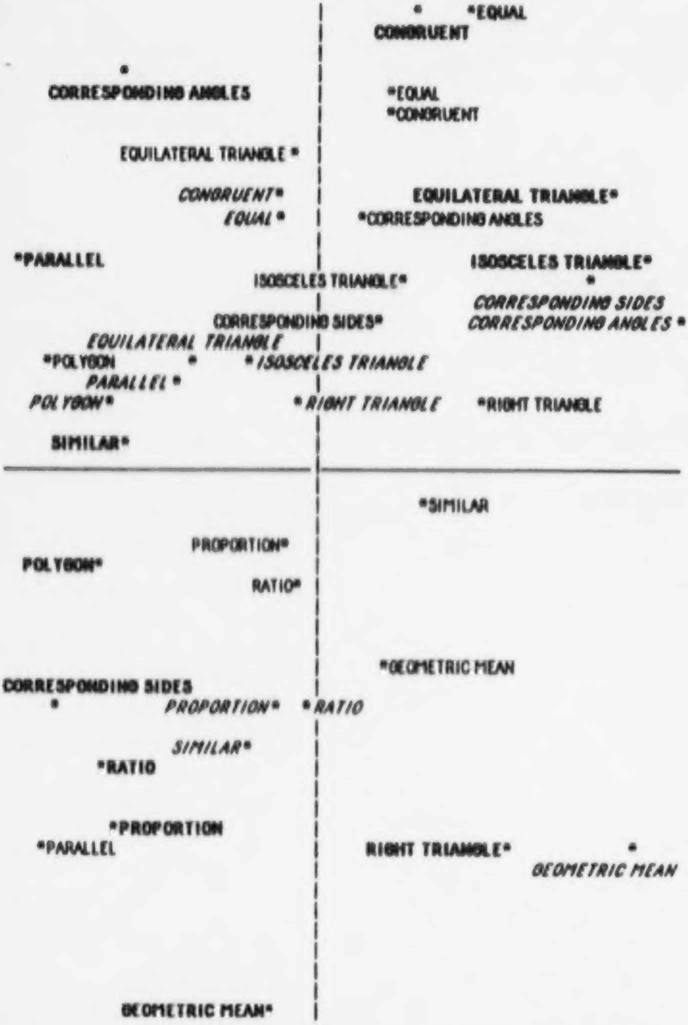

FIGURE 1. **EXPERT COGNITIVE REPRESENTATION**, PROTOTYPICAL FORMAL SUBJECT, *PROTOTYPICAL CONCRETE SUBJECT*

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SYMPOSIUM: LOGO AND MATHEMATICS LEARNING

Organized by Gerald Brazier, Pan American University

Presenters:

Will Watkins/Gerald Brazier, Pan American University
Patrick Thompson, Illinois State University
John Olive, Emory University
Pat Wilson, Ohio State University
Susan Blair/Thomas Kieren, University of Alberta

Reactors:

Ronald Wenger, University of Delaware
Tony Stavely, Keene State University

Since its inception, Logo has been touted as more than a computer language; it has been offered as a learning environment. Unlike the use of the computer as a support to instruction, the Logo environment has been presented as one in which the student is free to create his own world--not only to solve problems but to pose them as well. The appeal of this kind of promise for mathematics education is very strong.

Many questions arise when considering Logo and mathematics learning. Is the promise something that can be fulfilled? Are there special aspects of the Logo environment that create opportunities for investigating mathematics learning that other environments do not provide? This symposium will address a wide range of questions associated with Logo and will, with the contributions of other conference participants, spark discussions of many others.

Logo Geometry: Ego Syntonic?

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This study represents one test of a proposed theory which specifies levels of Logo use and relates such levels to the way van Hiele levels of geometric thinking. A motion geometry curriculum modelling this relationship was developed and validated. Two heterogeneous classes of students used the curriculum. A sample of 10 students were studies for 2 eighty minute periods a week for eleven weeks. An analysis of videotapes of these students working in pairs revealed the following: students who were at the Basic level in geometry were able to do sophisticated things in Logo mostly in direct mode. Naive procedure writing in Logo appeared to foster students knowledge of geometric properties. These students appeared able to transfer the knowledge to paper/pencil tasks and satisfactorally met the geometric objectives through the Logo use.

1. Background to the Study

It was the purpose of this research and is the purpose of this report to explore the ego syntonic nature of motion-geometry in a Logo environment. Ego syntonic geometry is taken here to mean geometry derived from and matched to the natural idea development of the learner. Thus, although experiences are provided, the kind of concept and level of language used is determined by the learner. So the question is, in what ways and to what extent is Turtle geometry ego syntonic (Papert, 1980)?

The van Hiele theory (Wirzup, 1976, Hoffer, 1983) presents one way of looking at geometry stratified in levels which match the perception/thinking process of the learner. Briefly stated these levels are: Basic Level: The objects are seen as wholes and are recognized by appearance alone. Level I: The objects are seen as carriers of properties which are not yet related. Level II: The objects are seen in terms of logical relationships among properties and among figures. Level III: The objects represent relationships which are deducible from an axiomatic system. This development occurs through learning, the process of which includes five phases: information, directed orientation, explanation, free orientation and integration. In these terms, Turtle geometry is ego syntonic if these levels and processes are evident in student Logo geometry activities.

For this to be the case it would seem that there should also be levels in Logo use. Martin, Paulsen and Prata (1984) have suggested such levels of Logo programming exist and Kieren (1984) has suggested the instructional structure for Logo used in Fig. 1 below.

Kieren and Olson (1983) saw a link between the van Hiele levels and levels of Logo use also illustrated in Fig. 1 below.

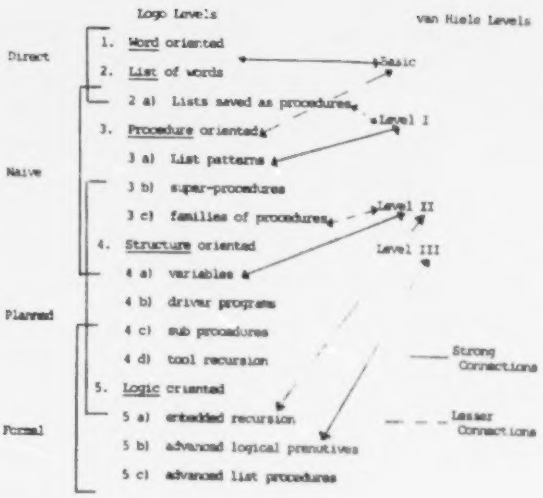
Logo Levels
van Hiele Levels
Direct
Naive
Planned
Formal
1. Word oriented
2. List of words
2 a) Lists saved as procedures
3. Procedure oriented
3 a) List patterns
3 b) super-procedures
3 c) families of procedures
4. Structure oriented
4 a) variables
4 b) driver programs
4 c) sub procedures
4 d) tool recursion
5. Logic oriented
5 a) embedded recursion
5 b) advanced logical primitives
5 c) advanced list procedures
Basic
Level I
Level II
Level III
Strong Connections
Lesser Connections

Figure 1

From this perspective geometry using Logo appears matchable with van Hiele levels and is thus ego syntonic in that sense.

To date there is some evidence of the levelled nature of Logo in the work of children. For example, Hillel (1984) observed that although children (8-9 years old) were encouraged to pre-plan Logo work after an initial period of introduction to Logo procedures most functioned at Levels 1 and 2 above, writing procedures mainly as a device to save lists which accomplished tasks. Even those who appeared to be at Level 3 and pre-planned procedures reverted to a lower level (screen debugging) when procedures didn't work as planned.

There have been related findings in research on van Hiele theory. For example, Burger (1985) notes that even high school students do not use properties to define geometric situations even though they might memorize adult or text given definitions or proofs.

To explore the ego syntonic nature of Logo based geometry and to test the relationship between Logo use and van Hiele levels theorized as signalling such nature a teaching study with the following purpose was done.

(1) Can one develop a set of experiences entailing Logo use at the Grade 7 level which allow for van Hiele levels and processes in student behaviour?

(2) In working in such an environment what are the Logo and geometric behaviours of the students?

(3) Are these behaviours consistent with the level theory in Figure 1 above?

(4) In what way does this curriculum facilitate movement from one level of Logo use or geometric thinking to the next? What are the evidences of such movement?

2. Research Procedures

2.1 Sample

Two heterogenous grade seven mathematics classes in an Edmonton junior high school were involved in the project. The work and progress of ten students, six in one class and four in the other was closely monitored, although all members of the classes were involved in the project.

2.2 The Curriculum

Following the principles for van Hiele geometry described in Hoffer (1983), a curriculum following the Grade objectives for motion geometry and involving Logo activities was devised. The following major topics in motion geometry were covered: translations, rotations and reflections. For each topic the following tasks were devised. An "Inquiry" phase involved computer demonstration and class discussion. In the second phase, "Turtle Tracking" the students used teacher designed procedures, created a motion in direct mode or debugged given procedures to complete the motion. The "Extensions" phase was intended for students to design and debug procedures related to the topic. They were assisted in writing procedures by the lists of commands recorded in the previous phase. "Turtle Excursions" for students at higher Logo or van Hiele levels was to extend students procedures to include variables or explore recursive procedures related to the topic. The final "Project" phase encouraged students to find their own solutions given some initial suggestions. The students were to use their knowledge gained through the previous four phases to write their own procedures. An intent of this phase was to integrate their previous knowledge gained. The motion geometry curriculum was taught for 2 eighty minute periods over an 11 week time span.

2.3 Observation and Data Analysis

One researcher served as a teacher observer in the experiment. Daily and at the end of the project videotapes of 10 students work were analyzed and coded to allow the following analysis: identifying behaviours at various van Hiele and Logo levels; document Logo processes used and difficulties encountered; identify examples of facilitation of geometric thinking if they exist; correlate geometric and Logo levels behaviourally.

3. Results

- The curriculum was validated three ways, through interactive design processes, by successful use in a class whose teacher was not previously Logo experienced, and in the teaching experiment itself.

- Subjects in the teaching experiment and cohort class learned geometry as assessed on a 35 item paper/pencil test. They also scored higher than five prior classes in the same school on a standard geometry test.

- Students were mainly at the Basic level of geometry and were moving to Level I. That is they were starting to independently find properties of motions but attempts at definitions were very imprecise.

- Similarly in Logo students in general moved from Direct mode to Naive programming through recognition of Direct mode patterns. They realized the value of writing preplanned procedures, but usually did not do so. Instead they normally worked on short lists of commands (debugging in Direct mode). Procedures usually came as concatenations of such efforts. Only one student independently recognized families of procedures and used variables in his programs.

- Logo procedure work (and even work with given geometric primitives) facilitated growth to Level I geometric thinking by giving students an opportunity to relate visual patterns in motions to verbal lists which formed the basis for properties.

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LOGO PROGRAMMING AND RELATIONAL LEARNING IN A GEOMETRIC MICROWORLD - IMPLICATIONS FOR INSTRUCTION

by

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Ninth grade students in an urban high school were taught Logo programming through Turtle graphics during an intensive six-week course. All students' interactions with Logo were captured in disk files and analyzed in terms of the SOLO/Skemp Synthesis to determine progressions through SOLO Learning Cycles and the appropriateness of the instructional sequence for generating relational learning. The analysis highlighted critical gaps in the instructional sequence and important pedagogical steps that need to occurr in order to generate relational learning.

INTRODUCTION

This research project was a pilot study for a larger investigation, now in progress, into students' understanding of geometric relationships: the Atlanta - Emory LOGO Project, which is supported by grants from the Apple Education Foundation and the National Science Foundation. The purpose of the pilot study was to investigate the potential of the LOGO computer language for generating relational learning cycles for students, in a geometric microworld, and to assess the appropriateness of the teaching methodology, sequence and content for generating relational learning.

A Logo teaching experiment was designed to help ninth grade students progress through the levels of the SOLO taxonomy (Biggs & Collis, 1980) in order to achieve a higher level of abstraction in their mathematical thinking. The mathematical focus of the instruction was on geometric relationships. The van Hiele model of geometric thought provided the rationale for this focus. The teaching methodology and curriculum ideas were based on a theory of relational learning cycles (Olive, 1983) which emerged from a synthesis of the SOLO taxonomy and Skemp's (1976) model of mathematical understanding within the context of Skemp's model of intelligence (1979).

METHODOLOGY

Twenty students were randomly chosen from an intact, ninth grade class of 39 students in an urban high school. Each student worked with a micro-computer in a lab situation for 18 days (two hours a day, three days per week for six weeks). The investigator taught the group, introducing the students to the micro-computer and the LOGO language through a series of "guided discovery" learning episodes.

Each student's interactions with LOGO were saved on disk files and analyzed in terms of the SOLO/Skemp synthesis. This analysis provided a picture of each student's developmental growth in the use of LOGO and helped to determine the appropriateness of the teaching methodology and curriculum ideas for generating relational learning cycles and helping students achieve a higher level of mathematical abstraction.

RESULTS

The results of the analysis indicate that for many students, the instructional sequence was too fast. There was not enough time for them to explore new programming ideas or to investigate the various geometric relationships before new ones were introduced. Consequently, their understanding of both the LOGO language and the geometric concepts was generally instrumental. However, for those students who were able to keep pace with the instruction, progression through SOLO learning cycles was evident. These students demonstrated a shift to a more abstract mode of functioning with the LOGO language and relational understanding of many of the geometric concepts that were introduced.

IMPLICATIONS FOR INSTRUCTION

The analysis of the data files also enabled the investigator to identify the gaps in the instructional sequence. These gaps highlighted the critical importance for introducing ideas at the appropriate SOLO level for individual students, for sequencing activities according to a SOLO cycle, and for encouraging reflection by the students on emerging relationships. The instructional sequence has been modified to better reflect these characteristics. An example of such a modification is given in the next section.

A RELATIONAL LEARNING SEQUENCE FOR POLYGONS

Critical gaps were discerned in the pilot curriculum where variable inputs to procedures were introduced and where a generalized procedure, which required three inputs, was introduced for the investigation of complex polygons. The following sequence was designed to fill these gaps and help develop relational understanding of both the use of variables in LOGO procedures <u>and</u> the mathematical relationships which emerged out of the investigation of complex polygons.

NOTE: This sequence begins at a point where students are comfortable with the definition of fixed procedures for generating individual geometric shapes, and with the use of REPEAT to generate regular polygons.

A. SHAPES

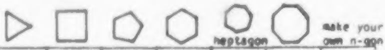
<u>Step 1: Generation of individual shape procedures</u>

Students define procedures for squares, equilateral triangles, pentagons, hexagons etc., using the REPEAT command.

EXAMPLE:
```
TO HEX
   REPEAT 6 [FD 50 RT 60]
END
```

<u>Step 2: Construction of a Shape Table</u>

Students complete the Shape Table (figure 1.) and construct the <u>relationship</u> between "angle turned" and "number of repeats" (Rule of 360).



Angle turned							
Number of repeats							
Product							

Figure 1.

Step 3: Varying the size of shapes with individual procedures

Students define a series of square procedures and triangle procedures which create different sized shapes. The size of the shape is indicated by the procedure name: e.g. SQ.20, SQ.30, SQ.40, SQ.50

```
EXAMPLE:  TO SQ.20                        TO SQ.30
             REPEAT 4 [FD 20 RT 90]          REPEAT 4 [FD 30 RT 90]
          END                             END
```

B. SHAPES WITH INPUTS

Step 1: Variable inputs to change the SIZE of shapes

Students compare each procedure in the series above to discover what is changing in each (the SIZE of the FD move), and what is staying the same (everything else). By analogy with the requirement that LOGO makes on FD to have an INPUT, the teacher can elicit from the students the idea that SQ could have an INPUT to tell it how big a square to draw. It is at this point that the LOGO syntax, to create variable inputs to procedures, can be meaningfully introduced. A variable name is used to pass the input from SQ to the FD command inside SQ:

```
TO SQ :SIZE
   REPEAT 4 [FD :SIZE RT 90]
END
```

Students can now generate variable procedures for changing the SIZE of each of their different shapes.

Step 2: Variable inputs to change the SHAPE of the polygon

After a great deal of exploration and building with the variable procedures for each shape, a discussion comparing similarities and differences among the different shape procedures, and the relationship between "angle turned" and "number of REPEATS" (Rule of 360) inherent in each procedure, can lead to a more generalized form of each shape procedure, using this relationship:

```
EXAMPLE:  TO HEX :SIZE
             REPEAT 6 [FD :SIZE RT (360 / 6)]
          END
```

The only difference now between each procedure is the number of REPEATS.

A discussion should elicit the idea of using a <u>variable</u> to stand for t number of REPEATS. It would be appropriate to generate a <u>shape changi</u> procedure of a <u>fixed size</u> at this point:

```
TO FPOLY :N                              (fixed size polygon
   REPEAT :N [FD 50 RT (360 / :N)]
END
```

Step 3: Variable inputs to change both SHAPE and SIZE

Students will want to change the SIZE of shapes created with FPOL discussion should elicit the idea of having TWO INPUTS:

```
TO RPOLY :N :S                           (regular polygon)
   REPEAT :N [FD :S RT (360 / :N)]
END
```

C. SHAPES WITHOUT RESTRICTIONS

Step 1: Removing the restriction on the angle relationship

RPOLY <u>always</u> produces a simple, closed, regular polygon because c built-in relationship between "angle turned" and "number of REPEATS." discussion of what would happen if we lifted that restriction so that <u>input any angle</u> should precede the introduction of the more generaliz input procedure:

```
TO APOLY :N :S :A
   REPEAT :N [FD :S RT :A]
END
```

Step 2: Explorations with APOLY

Students can now explore a wider class of geometric figures, inc open figures and complex polygons. An investigation of inputs to APOl produce closed, complex polygons (star shapes) can lead to an underst the highly complex relationship between :N and :A needed to produce a polygon, and the creation of a generalized procedure which embodies t relationship:

```
TO MPOLY :N :S :M
   REPEAT :N [FD :S RT (:M * 360 / :N)]
END
```

MPOLY produces an N-pointed star when :M (modulo :N) and :N have no c factor, and :M is greater than one.

FINAL COMMENT

The above instructional sequence progresses through SOLO levels (Unistructural through Relating) for a particular type of LOGO object (Fixed Procedure, Variable Procedure, Generalized Procedure) before introducing the more complex LOGO object. Each new LOGO object is introduced after students have had the opportunity to *reflect* on what they have been doing with existing LOGO objects and the relationships they have discovered using those objects. These two sequencing elements emerged as key elements for generating relational learning cycles for students, and helping them achieve Extended Abstract SOLO responses, indicative of higher levels of mathematical thinking.

The results also demonstrate the enormous potential for process analysis of data provided by the "dribble" file technology. The ability to visually recreate every step a student takes when working on a problem brings us closer to being able to directly observe an individual's cognitive processes.

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UNDERSTANDING RECURSION: PROCESS ≈ OBJECT

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It is hypothesized that to recognize a computation as requiring recursion, students must conceptualize a reciprocal relationship between processes and their resulting objects. An example is given, along with a discussion of the role of recursion within a mathematics curriculum.

Ask any instructor of Pascal, Logo, LISP, data structures, or algorithms to name three topics that students find most difficult. Most probably his or her list will include *recursion*. In this brief paper I will propose an hypothesis for explaining students' extreme difficulty with recursion, and will justify the importance of recursion's place in a mathematics curriculum.

First, let us ensure a common vocabulary. The term *recursive process* will mean any process that employs itself as a subprocess. The term *recursive object* will mean any object which contains an instance of itself as a component. *Recursion* will mean the class of recursive processes and recursive objects.

AN HYPOTHESIS

The distinction I have drawn between recursive processes and recursive objects is essential to formulate my hypothesis concerning students' difficulties with recursion. The hypothesis is this: to be able to recognize a problem solution as one requiring a recursive process, students must formulate their solution as a recursive object. Conversely, to recognize an object as having a recursive structure, they must formulate their description of it so that it is the result of a recursive process. That is to say, students must approach a problem with the anticipation that every object is the result of a process and every process results in an object. An example will illustrate this point.[1]

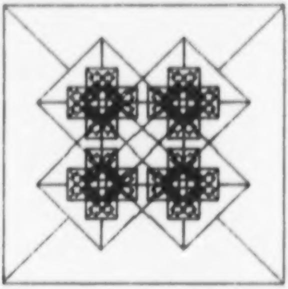
---

[1] The examples are written in ExperLogo, which in these examples is identical to Apple Logo.

AN EXAMPLE

The figure below is one I regularly give my introductory Logo students with the intention that they write a procedure to construct a class of figures of which this is but one example.[2] Here is the kind of analysis that corresponds to this paper's hypothesis:

1. Name the class of objects: (Of those names that have been suggested, ASHTRAY is my favorite).
2. Describe an ASHTRAY: An ASHTRAY of order *n* and of size *s* is a square-with-tails with an ASHTRAY of order *n-1* and of size *s/3* at the tip of each tail.
3. State the minimal case: An ASHTRAY of order *0* is a point.



The description of the class ASHTRAY not only describes the class, it suggests a process by which to construct one. Since, in Logo, graphics is created by moving a turtle we also need to specify the relationship between the turtle and the to-be-drawn figure and to specify the relationship between the turtle's beginning and ending states when making an ASHTRAY.

4. The relationship between the turtle's initial state and a to-be-constructed ASHTRAY is that the turtle is, from its perspective, in the middle of the bottom side pointing perpendicularly toward the opposite side of the square-with-tails. (Other relationships are possible; this one merely turns out to be convenient.)

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[2] It actually requires a sequence of pictures to veridically suggest that the class has a recursive structure. To conserve space, I give a sequence of length one.

5. The effect upon the turtle of making an ASHTRAY is nil. That is, the turtle's beginning and ending states are identical. (This is merely a convenient assumption; any relationship between beginning and ending state is possible).

Notes 1 through 5 can be thought of as design specifications for making an ASHTRAY. To write the corresponding procedure, a student need only implement the description of the class as given in notes 1 through 3, keeping in mind the relationships specified by notes 4 and 5. However, to implement the description of an ASHTRAY *one must anticipate that the process one is describing produces an object of the described class, and that to obtain an object in the class one invokes the name of the process.* To make an ASHTRAY of order *n* and of size *s*, we will invoke the process named ASH.

```
TO ASH :N :S
  IF :N=0 [STOP]                  An ASHTRAY of order 0 is a point (recall note 3).
  SQUARE.WITH.TAILS :N :S         SQUARE-WITH-TAILS will put an ASHTRAY at the tip of
                                  each tail (recall note 2).
END

TO SQUARE.WITH.TAILS :N :S
  REPEAT 4 [ LT 90 FD :S/2        Go to the end of the immediately-left tail (recall note
             RT 135 FD :S/3       4).

             ASH :N-1 :S/3        Make an ASHTRAY at the end of the tail (recall note 2).
                                  Turtle ends where it now sits (recall note 5).

             BK :S/3 LT 45
             FD :S/2 RT 90 ]3     Go to the middle of the next side.
END
```

The decision to write ASH :N-1 :S/3 in line 4 of SQUARE.WITH.TAILS is where it is essential to relate process and object as two sides of a coin. Many students, instead of writing ASH :N-1 :S/3, will begin to write LT 90 FD :S/2 RT 135 .... which is the beginning of another SQUARE.WITH.TAILS. That is, they become trapped in the *process* of constructing an ASHTRAY, do not recognize that what is required at that point is another *object* called an ASHTRAY, and that any ASHTRAY can be created by invoking ASH.

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[3] In most versions of Logo, the commands in the REPEATed list must be typed as one logical line (i.e., without carriage returns).

The necessity of relating process and object applies equally well to problems of writing recursive functions that operate upon data. However, students generally find writing recursive functions more difficult than writing recursive graphics. Apparently, when writing recursive graphics it is often sufficient to use an image of the finished product as a stimulant to cue themselves as to when to make recursive calls. When writing recursive functions, it is generally insufficient to imagine only the finished product (the function's output). Students must also reflect the data's structure in the function's structure. This is what Touretzky (1983) calls *structured recursion*. An example of reflecting a datum's structure within a function for processing it will be given in the presentation.

THE HYPOTHESIS REFORMULATED

To be able to write recursive procedures or functions (as distinct from merely reading a recursive procedure or function written by someone else), students must first describe the object to be created by the procedure in a way that reflects its recursive structure. They then can use that description as a guide for writing the procedure, keeping in mind that whenever they require an object of a particular class they invoke the name of the procedure that creates it, regardless of whether or not (at the time they invoke the name) the procedure has been completely defined. The cognitive prerequisites for this ability amount to a mindset, or belief system:

1. Any process produces an object.[4]
2. One obtains an object of a particular class by invoking the name of the process that creates it.
3. One can name (and hence invoke) a process before the process has been defined (with the intention that it will be defined eventually).

RECURSION IN MATHEMATICAL UNDERSTANDING

The kind of object oriented thinking discussed above permeates theories of mathematical understanding. Skemp (1979) discussed a two-level model of mathematical thinking: at the lower level, one thinks by doing. At the higher level, one thinks *about* doing. Freudenthal

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[4] This includes "nonterminating" processes, which allows the set of natural numbers to be considered as an object. However, I would imagine that in practice most intended objects result from terminating processes.

(1972) discussed mathematical development in terms of progressively higher levels of object construction. Piaget's constructs of reflection and reflexion (Piaget & Inhelder, 1969) addressed the distinction between actions and represented actions. In each case, descriptions of intellectual advancement involve hypothesizing that what is process at one level becomes object at another level. Here, in students' study of recursion, we have the opportunity to provoke students' into making the relationship between process and object explicit to themselves. One must be cautioned, however, by the empirical question as to whether or not such an awareness will actually assist students in their mathematical development.

RECURSION IN THE CURRICULUM

The school curriculum is rife with opportunities for casting mathematical content recursively. One example is given here. It defines a "grammar" for integers and integer operations (Dreyfus & Thompson, 1985). Here, the semantics of an integer is:

number	Do *number* steps in your current direction
-number	Turn around, do *number*, turn back around

The grammar for integers is:

1. A whole number is a number.
2. The negative of a number is a number.
3. The composition of two numbers is a number (one composes numbers by doing them consecutively).
4. The representation of a composition is equivalent to a number.
5. An operation defined as a composition of numbers is equivalent to a number.

The recursive property of the system (grammar and semantics) manifests itself when one evaluates expressions, as in **-[-70 30]**, which denotes the negative of the composition of **-70** and **30**. Our research suggests that for students to employ a rule of substitution when evaluating expressions, they must construct the distinction between process and object, as was hypothesized earlier in this paper for writing recursive procedures (Dreyfus & Thompson, 1985; Thompson & Dreyfus, 1985). Other examples can be found in topics ranging from whole number numeration to mathematical analysis.

CONCLUSION

It should be noted that the focus in this paper was explaining students' difficulties in *creating* recursive processes and objects. This is quite different from studies that focus on students' abilities to recognize already-written procedures as being recursive (cf. Kurland & Pea, no date) or their abilities to write iterative processes under the guise of recursion ("tail-end" recursion; cf., Anzai & Uesato, 1982). The ability to create recursive processes and objects is much more difficult to cultivate than abilities to recognize "recursiveness" in already-written procedures, but at the same time once attained is much more useful.

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THE CHAIN RULE IN THE LOGO ENVIRONMENT

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Abstract

Two university students were introduced to the list-processing primitives of Logo and were presented with the task of producing a symbolic differentiation procedure. Under the non-directive direction of the two investigators, they were able to complete the task and refine their own thinking about composite functions and the chain rule.

INTRODUCTION

Logo has been highly touted as the computing environment par excellence for mathematics learning. Papert's (1980) enthusiasm has been infectious but a body of research investigating Logo and mathematics learning has been slow to develop.

"An environment in which students can gain control over their own learning," is the way in which Logo is described. Such a description has particular appeal to mathematics education reseRachers who are directed by a constructivist point of view of learning (von Glasersfeld, 1983). Opportunities to truly construct new knowledge in an overt, conscious way are very rare in school settings. Rarer still are opportunites to study such learner activity in a scientific way. This investigation represents the barest beginnings of an effort to use the Logo environment to study a learner's construction of knowledge. The content chosen was symbolic differentiation--material dominated by rules and form.

PROCEDURES

The two students who participated in the investigation were volunteers from two different calculus classes. The first student, Vicki, was enrolled in the one semester survey of calculus for business majors. It was her first experience with the calculus. The second student, Neal, was enrolled in the second semester of the calculus sequence for science majors. Each of the classes was presented with the opportunity to spend an hour each day "doing some calculus on the computer." Several students expressed interest but Vicki and Neal were the only ones who followed through on the complete program. No attempt was made to gather background information on the two students other than having them describe their understanding of the mechanics of the chain rule. Neal was well-acquainted with the material and Vicki had been taught the chain rule (in the context of the power rule) just before the first session on the computer.

The first phase of the investigation consisted of an introduction to the MIT version of Logo for the Apple with particular attention to a subset of the list-processing primitives--namely, FIRST, LAST, BUTFIRST, BUTLAST, FPUT, LPUT, and LIST. Much of the first session was taken up with familiarization activities, some graphics, and an introduction to creating procedures. In the second session the students began their investigation of the list-processing primitives by creating their own procedures to solve certain kinds of standard problems--find the second element in a list, determine whether a given element is in a list, and so forth. The two investigators worked with the students individually and as a pair in developing their understanding of Logo's handling of inputs and its conditional IF construct. By the end of the second session the students were writing short procedures employing the list-processing primitives described above. The third session was spent solidifying these ideas by presenting the students with some more challenges along the same line.

The second phase consisted of developing a scheme for representing functions in the Logo environment. The investigators presented these three examples for consideration: [PW X 2], [P 3 X], [S X 7]. These were representations for X^2, 3X, and X + 7, respectively. At this point the students were given exercises in translating standard composite functions into this new form--expressions like $(3X + 8)^4$ to be represented as [PW [S [P 3X] 8] 4]. At the end of this session, the task of writing a procedure to produce the derivative of a function was proposed to the students. Since both students' work in the calculus to this point had been dominated by power functions, they were immediately drawn to the task of differentiating X^n. Each of the students by the end of the next session had created the following linked procedures (with minor differences):

```
TO PWR :F
   OP ( LIST "P LAST :F DER :F )
END
TO DER :F
   OP LPUT (LAST :F) -1 BL :F
END
```

At this point the investigators posed the question of how to incorporate the chain rule into the scheme and then after discussion proposed the following skeleton master procedure:

```
TO DX :F
   IF :F = "X OP 1
   IF NUMBER? :F OP 0
   IF FIRST :F = "PW OP DPWR :F
   . . .
   OP LIST "DX :F
END
```

This skeleton required building the procedure DPWR from what had been done in PWR as follows:

```
TO DPWR :F
   OP ( LIST "P PWR :F DX FIRST BF :F )
END
```

The recursive nature of the procedure DX is clear and yet was not emphasized in any way by the investigators--the students knew that the chain rule required a product so simply wrote their implementation of the chain rule that way.

The third and final phase was simply the students' carrying out the task of filling out the skeleton procedure DX to incorporate as many different functions (sum, product, etc.) as they could. It was in this phase that the monitoring of their work become the major focus of the investigation.

RESULTS

Detailed records of the students' procedures in various stages of completion are available from the authors. These records capture some of the flavor of the students' experirence but there were many other aspects of the investigation that can only, at present anyway, be observed in a very informal and imprecise way. There is no question of the rapidity with which both students were able to produce the subprocedures necessary to symbolically differentiate sums, products, quotients, exponentials, and logarithms. In observing their work, the pattern of decomposition of large task to smaller tasks was evident at every turn. Both students very quickly developed a "case-study" approach by which they made the machine mimic their own thought processes in praticular elaborated examples. They recognized the necessity for having a sufficiently complicated prototype to work with as they taught the machine to think like they thought. Vicki, though less experienced mathematically, developed an extremely efficient algorithm by which she developed the procedures--a fascinating bit of meta-cognition.

Each of the students become extremely adept at decomposing elaborate functions and in creating standard notation for results created by the infix notation of their procedures. In fact, Neal spent time at the end of the investigation writing procedures to simplify expressions like [S X 0] and [P 3 1] to X and 3, respectively. Both students were successful in their classroom work within which the ten day investigation took place but there is no way to attribute that to their Logo experience. In a formal way, very little can be noted in the way of results, yet informally it was seen that each of students engaged in thinking about mathematics in a way they had not before.

CONCLUSION

In the spirit of a bare beginning, the investigation reveals some of the potential of the Logo environment for creating a workplace for mathematics learners. With very minimal start up cost students can be working on significant tasks that allow them to reflect on their own mathematical knowledge. How they proceed needs to be monitored more carefully and needs to be correlated more carefully with what else we can know about their mathematical knowledge. The potential for using the Logo environment for a laboratory both for students and for those studying students seems very great.

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LOGO: A POWERFUL RESEARCH TOOL

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Current research using Logo is discussed with a focus on the special opportunities presented by Logo.

"Do Not Ask What Logo Can Do To People, But What People Can Do With Logo" was the borrowed challenge that Papert (1985) presented to researchers at the Logo 85 Conference. Papert's comments were a reaction to the criticism of Logo resulting from studies finding little or no significant difference in Logo treatments (Bank Street College, 1983-84; University of Edinburgh, 1970- ; Brookline). He claimed current empirical research asks the wrong questions. Insisting that a scientific paradigm is not appropriate, Papert strongly opposed using a treatment study to investigate the value of Logo. He explained that one should not try to evaluate the effect of Logo, but should report how it was used and the consequences.

Papert (1980) claims that Logo microworlds can be created that are incubators for knowledge. This is supported by Leron's explanation that most of the students he has worked with seem to have gained "some sort of vague, partial understanding of many powerful ideas" (1985a, p. 32). The idea of a knowledge incubator where partial understandings of powerful ideas are developing is an exciting laboratory situation for a researcher! The following sections discuss the research opportunities in a Logo environment.

LOGO PROVIDES A WINDOW

Logo can be used as a research tool that provides a window into a student's complex world of thinking. In this sense, the researcher is not interested in studying Logo but is interested in using Logo. The Logo task (structured or unstructured) provides an opportunity for the researcher to learn more about what a student is thinking by observing and interacting with the student.

Papert (1985) offers an example of studying students' styles of working. Student A knew exactly what he wanted to do and set out to achieve the goal. The student continually made modifications but continued to work toward the goal. Student B messed around. He tinkered with one idea, jumped to another idea and continued to explore ideas as they occurred. Eventually select ideas were integrated into a project. Logo exploration allowed the students the freedom to exhibit their preferred style of working.

Using a moderately structured Logo task, Wilson (1984) found upper elementary school students were able to share their understanding of algebraic ideas such as variable and iteration. The Logo task provided a source of examples and vocabulary for the students to use as they expressed their ideas. Given a task to create a regular polygon with 6 sides, a fourth grader proclaimed that "The numbers have to fit!". Her vocabulary did not permit her to *explain verbally* that the angles in a regular polygon are a function of the number of sides. She could *display* an example where the numbers fit (a hexagon) and where the numbers did not fit (a open figure with 6 equal sides).

LOGO PROVIDES A PROGRAMMING BACKGROUND

Noss (1985) is studying how children with extensive Logo programming experience construct mathematical meaning. Noss notes that previous research has focused on *how* children learn Logo and *what* mathematical knowledge has been learned by using Logo. He is interested in a third question of what mathematics children *can* learn via Logo. He uses a series of "solve-aloud" problem interviews which often ask eleven year-old children how they would explain their ideas to a first grader. The Logo environment permits children to construct their own notation and to formalize their own rules. He has found that students often use programming vocabulary. The student is in control of how the ideas are expressed rather than trying to interpret an instrument with conventional notation.

Logo provides structured programming with an emphasis on procedures. In a carefully documented study at Concordia University, Erlwanger and Barfurth (1985) are using the idea of a Logo procedure to link mathematics and programming concepts. The students begin with concrete materials (i.e., building blocks, popsicle sticks) in order to investigate ideas associated with distance, length, direction, angles and shapes. Next they use paper and pencil to write programs and finally they use the computer to see if the screen image matches their mental image.

LOGO OFFERS A VARIETY OF DATA FOR ANALYSIS

Olive and Scally (1985) are using dribble files that record all student interaction with the computer. Several researchers have used videotapes of children working, screen output, and paper and pencil activities to supplement their observations (Erlwanger & Barfurth, 1985; Noss, 1985; Hillel, 1985).

The opportunities for examining how ideas are developed are exciting, but words of caution are necessary. Anecdotes are a useful way to explain or report student activity; however, researchers must be careful not to report only favorable or idealized anecdotes. Leron (1985b) adds that the researcher must not confuse the mathematics that the researcher sees with the mathematics that a student sees in a Logo situation. Erlwanger and Barfurth (1985) convincingly argue for careful documentation of the setting, organization and the procedures in any description of results.

Logo offers a powerful tool for investigating the development of mathematical ideas.

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